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A Course in Minimal Surfaces

极小曲面教程

Tobias Holck Colding
William P. Minicozzi II

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

The motivation for these lecture notes on minimal surfaces is to have a treatment that begins with almost no prerequisites and ends up with current research topics. We touch upon some of the applications to other fields including low dimensional topology, general relativity, and materials science.

Minimal surfaces date back to Euler and Lagrange and the beginning of the calculus of variations. Many of the techniques developed have played key roles in geometry and partial differential equations. Examples include monotonicity and tangent cone analysis originating in the regularity theory for minimal surfaces, estimates for nonlinear equations based on the maximum principle arising in Bernstein's classical work, and even Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces.

The only prerequisites needed for this book are a basic knowledge of Riemannian geometry and some familiarity with the maximum principle. Of the various ways of approaching minimal surfaces (from complex analysis, PDE, or geometric measure theory), we have chosen to focus on the PDE aspects of the theory.

In Chapter 1, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the first chapter is on the basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the next to last section of Chapter 1, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface. In the last section, we introduce multi-valued minimal graphs which will play a major

role later when we discuss results from [**CM3**]–[**CM7**]. We will also give a local example, from [**CM18**], of spiraling minimal surfaces (like the helicoid) that can be decomposed into multi-valued graphs but where the rate of spiraling is far from constant.

Chapter 2 deals with generalizations of the Bernstein theorem. We begin the chapter by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we discuss various applications of this inequality. The first application is a theorem of Choi and Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding and Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen, Simon and Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We introduce a notion of "almost stabilility" that plays a crucial role in understanding embedded surfaces. Next, we return to multi-valued minimal graphs and prove an important result from [**CM3**] which states that the separation grows sublinearly if the multi-valued graph has enough sheets. We close the chapter with a discussion of minimal cones in Euclidean space and the relationship to the Bernstein theorem.

We start Chapter 3 by introducing stationary varifolds as a generalization of classical minimal surfaces. We next prove the Sobolev inequality of Michael and Simon. After that, we prove a generalization, due to Colding and Minicozzi, of the Bernstein theorem for minimal surfaces discussed in the preceding chapter. Namely, following [**CM6**], we will show in Chapter 3 that, in fact, a bound on the density gives an upper bound for the smallest affine subspace that the minimal surface lies in. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally, properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space. Finally, in the last section, we introduce another notion of weak convergence (called bubble convergence) that was developed to explain the bubbling phenomenon that occurs in conformally invariant problems, including two-dimensional harmonic maps and J-holomorphic curves. We will show that bubble convergence implies varifold convergence.

Chapter 4 begins with the solution to the classical Plateau problem for maps from surfaces. There is a close connection between energy and area in dimension two and the main issue is to understand the lack of compactness, called "bubbling", for maps with bounded energy. The first three sections cover the basic existence results for the Dirichlet and Plateau problems for maps from disks, while the fourth section discusses branch points. After that, we turn to the existence of harmonic maps from the two-sphere, following the approach by Sacks and Uhlenbeck, [**SaUh**], of first minimizing a perturbed energy functional and then taking the limit as the perturbation goes to zero.

In Chapter 5, we use a very general argument, whose basic idea goes back to H.A. Schwarz in the 1870s and G.D. Birkhoff in the 1910s, to find minimal spheres on any sphere. The treatment will follow the papers [**CM27**] and [**CM28**] where some of the existence results were new. The idea of both Schwarz and Birkhoff was to use a min-max argument to show existence of critical point for variational problems. This allows us, in particular, to produce minimal surfaces that are not stable. In the min-max construction of minimal surfaces, one sweeps out the manifold by surfaces keeping track of the areas of the slices of the sweepout. One then tries to extract a convergent sequence of maximal slices for which the area of the maximal slice converges to the infimum of the maximal slices of all sweepouts.

Chapter 6 focuses on the regularity of classical solutions to the Plateau problem. After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

In Chapter 7, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). We then prove the results of Hersch, and Choi and Wang. Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that by results of Choi and Wang and Yang and Yau such minimal surfaces have uniform area bounds. We then prove the positive mass theorem of Schoen and Yau. In the last section, we prove the Colding-Minicozzi finite extinction theorem for Ricci flow on a homotopy three-sphere.

Finally, in Chapter 8, we will present some recent results on embedded minimal surfaces in $\mathbb{R}^3$. We begin with a local result from [**CM4**] which shows that an embedded minimal disk is either graphical or, on a slightly larger scale, contains a double-spiral staircase. We also state the one-sided curvature estimate from [**CM6**]. This theorem roughly asserts that an embedded minimal disk in $\mathbb{R}^3$, that lies on one side of a plane and comes close to it is a graph over the plane. The novel thing about this estimate is that it does not require any a priori bound unlike all of the classical results for minimal surfaces discussed in the previous chapters of this book. These results are the starting point for the structure of embedded minimal surfaces obtained in the series of papers [**CM3**], [**CM4**], [**CM5**], and [**CM6**]. We then describe an application (from [**CM10**]) of the one-sided curvature estimate to prove the Generalized Nitsche Conjecture (proven originally by P. Collin, [**Co**]). After this, we turn to the resolution of the Calabi-Yau conjectures for embedded surfaces from [**CM24**]. Finally, we describe the main structure theorems from [**CM7**] for embedded minimal surfaces with finite genus and several recent uniqueness results that have relied upon the structure theory of [**CM3**]–[**CM7**].

At the end of the book, problems and exercises are given.

It is a pleasure to thank Matthias Weber for creating some of the figures for this book and Sergei Gelfand for his persistence and encouragement.

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Contents

Chapter 1

The Beginning of the Theory

In this chapter, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the chapter is on some basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the next to last section, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface. In the last section, we introduce multi-valued minimal graphs which will play a major role later when we discuss results from [**CM3**]–[**CM7**]. We will also give a local example, from [**CM18**], of spiraling minimal surfaces (like the helicoid) that can be decomposed into multi-valued graphs, but where the rate of spiraling is far from constant.

1. The Minimal Surface Equation and Minimal Submanifolds

1.1. Graphs and the minimal surface equation. Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u

$$\text{Graph}_u = \{(x, y, u(x, y)) \mid (x, y) \in \Omega\}\,. \tag{1.1}$$

Then the area is

$$\begin{aligned}\text{Area}(\text{Graph}_u) &= \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| \\ &= \int_\Omega \sqrt{1 + u_x^2 + u_y^2} = \int_\Omega \sqrt{1 + |\nabla u|^2}\,,\end{aligned} \tag{1.2}$$

and the (upward pointing) unit normal is

$$N = \frac{(1,0,u_x)\times(0,1,u_y)}{|(1,0,u_x)\times(0,1,u_y)|} = \frac{(-u_x,-u_y,1)}{\sqrt{1+|\nabla u|^2}}\,. \tag{1.3}$$

Applying this to a one-parameter family of graphs $\mathrm{Graph}_{u+t\eta}$, where $\eta|\partial\Omega = 0$ and t is the parameter, we get that

$$\mathrm{Area}(\mathrm{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1+|\nabla u + t\,\nabla\eta|^2}\,; \tag{1.4}$$

hence, the directional derivative of the area functional on graphs at u in the direction η is

$$\begin{aligned} \frac{d}{dt}_{t=0}\mathrm{Area}(\mathrm{Graph}_{u+t\eta}) &= \int_\Omega \frac{\langle \nabla u\,,\nabla\eta\rangle}{\sqrt{1+|\nabla u|^2}} \\ &= -\int_\Omega \eta\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \end{aligned} \tag{1.5}$$

Therefore, the graph of u is a critical point for the area functional if u satisfies the divergence form equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0\,. \tag{1.6}$$

Equation (1.6) is the divergence form of the *minimal surface equation* and can alternatively be written as

$$\begin{aligned} 0 &= \left(1+|\nabla u|^2\right)^{\frac{3}{2}}\left[\left(\frac{u_x}{\sqrt{1+|\nabla u|^2}}\right)_x + \left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y\right] \\ &= (1+u_y^2)\,u_{xx} + (1+u_x^2)\,u_{yy} - 2\,u_x\,u_y\,u_{xy}\,. \end{aligned} \tag{1.7}$$

Next we want to show that the graph of a function on Ω satisfying the minimal surface equation is not just a critical point for the area functional but is actually area-minimizing amongst surfaces in the cylinder $\Omega\times\mathbb{R}\subset\mathbb{R}^3$. Let ω be the two-form on $\Omega\times\mathbb{R}$ given by that for X, $Y\in\mathbb{R}^3$,

$$\omega(X,Y) = \det(X,Y,N)\,, \tag{1.8}$$

where

$$N = \frac{(-u_x,-u_y,1)}{\sqrt{1+|\nabla u|^2}}\,. \tag{1.9}$$

Observe that

$$\omega\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\frac{1}{\sqrt{1+|\nabla u|^2}}\begin{vmatrix}1 & 0 & -u_x\\ 0 & 1 & -u_y\\ 0 & 0 & 1\end{vmatrix}=\frac{1}{\sqrt{1+|\nabla u|^2}}\,, \tag{1.10}$$

$$\omega\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=\frac{1}{\sqrt{1+|\nabla u|^2}}\begin{vmatrix}0 & 0 & -u_x\\ 1 & 0 & -u_y\\ 0 & 1 & 1\end{vmatrix}=\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\,, \tag{1.11}$$

and

$$\omega\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)=\frac{1}{\sqrt{1+|\nabla u|^2}}\begin{vmatrix}1 & 0 & -u_x\\ 0 & 0 & -u_y\\ 0 & 1 & 1\end{vmatrix}=\frac{u_y}{\sqrt{1+|\nabla u|^2}}\,. \tag{1.12}$$

Hence

$$\omega=\frac{dx\wedge dy-u_x\,dy\wedge dz-u_y\,dz\wedge dx}{\sqrt{1+|\nabla u|^2}} \tag{1.13}$$

and

$$d\omega=\frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\right)+\frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}}\right)=0\,, \tag{1.14}$$

since u satisfies the minimal surface equation. In sum, the form ω is closed and, given any orthogonal unit vectors X and Y at a point (x,y,z),

$$|\omega(X,Y)|\leq 1\,, \tag{1.15}$$

where equality holds if and only if

$$X,Y\subset T_{(x,y,u(x,y))}\,\mathrm{Graph}_u\,. \tag{1.16}$$

Such a form ω is called a *calibration* and it can be used to show that Graph_u is area-minimizing:

Lemma 1.1. *If $u:\Omega\to\mathbb{R}$ satisfies the minimal surface equation and $\Sigma\subset\Omega\times\mathbb{R}$ is any other surface with $\partial\Sigma=\partial\,\mathrm{Graph}_u$, then*

$$\mathrm{Area}(\mathrm{Graph}_u)\leq\mathrm{Area}(\Sigma)\,. \tag{1.17}$$

Proof. Since ω is a closed form and Graph_u and Σ are homologous, Stokes' theorem gives

$$\int_{\mathrm{Graph}_u} \omega = \int_{\Sigma} \omega \,. \tag{1.18}$$

Combining this with (1.15) and (1.16) gives

$$\mathrm{Area}(\mathrm{Graph}_u) = \int_{\mathrm{Graph}_u} \omega = \int_{\Sigma} \omega \leq \mathrm{Area}(\Sigma) \,. \tag{1.19}$$ □

Throughout, we will use $B_r(x)$ to denote the ball in $\mathbb{R}^3$ (or in $\mathbb{R}^{n+1}$) with radius r and center x; we will often write B_r for the ball $B_r(0)$. We will use D_r for the disk of radius r in $\mathbb{R}^2$ centered at 0.

Corollary 1.2. *If $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation and $D_r \subset \Omega$, then*

$$\mathrm{Area}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Area}(\mathbf{S}^2)}{2}\, r^2 = 2\pi\, r^2 \,. \tag{1.20}$$

Proof. Since $\partial B_r \cap \mathrm{Graph}_u$ divides ∂B_r into two components at least one of which has area at most equal to $(\mathrm{Area}(\mathbf{S}^2)/2)\, r^2$, (1.17) gives (1.20). □

If the domain Ω is convex, the minimal graph is absolutely area-minimizing. To see this, observe first that for a convex set Ω the nearest point projection $P : \mathbb{R}^3 \to \Omega \times \mathbb{R}$ is a distance nonincreasing Lipschitz map that is equal to the identity on $\Omega \times \mathbb{R}$. If $\Sigma \subset \mathbb{R}^3$ is any other surface with $\partial\Sigma = \partial\, \mathrm{Graph}_u$, then $\Sigma' = P(\Sigma)$ has $\mathrm{Area}(\Sigma') \leq \mathrm{Area}(\Sigma)$. Applying (1.17) to Σ', we see that $\mathrm{Area}(\mathrm{Graph}_u) \leq \mathrm{Area}(\Sigma')$ and the claim follows.

Very similar calculations to the ones above show that if $\Omega \subset \mathbb{R}^{n-1}$ and $u : \Omega \to \mathbb{R}$ is a C^2 function, then the graph of u is a critical point for the area functional if and only if u satisfies the equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \,. \tag{1.21}$$

Moreover, as in (1.17), if u satisfies (1.21), then the graph of u is actually area-minimizing. Consequently, as in (1.20), if Ω contains a ball of radius r, then

$$\mathrm{Vol}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{2}\, r^{n-1} \,. \tag{1.22}$$

1.2. The geometry of submanifolds. We could also have looked more generally for a k-dimensional submanifold Σ possibly with boundary and sitting inside some Riemannian manifold M (with metric g and covariant derivative ∇) and which is a critical point for the area functional.

In the following, if X is a vector field on $\Sigma \subset M$, then we let X^T and X^N denote the tangential and normal components, respectively. The covariant derivative ∇ on M then induces a covariant derivative ∇_Σ on Σ and second fundamental form A of Σ. That is, the induced covariant derivative ∇_Σ is given by

$$\nabla_\Sigma = (\nabla)^T \,, \tag{1.23}$$

and the vector-valued bilinear form A on Σ is given for X, $Y \in T_x\Sigma$ by

$$A(X,Y) = (\nabla_X Y)^N \,. \tag{1.24}$$

Since the bracket of tangential vector fields is again a tangential vector field, it is easy to see that A is symmetric, i.e., $A(X,Y) = A(Y,X)$. Observe that

$$\begin{aligned} \sum_{\ell=1}^{n-k} g(A(X,Y), N_\ell)\, N_\ell &= \sum_{\ell=1}^{n-k} g(\nabla_X Y, N_\ell)\, N_\ell \\ &= -\sum_{\ell=1}^{n-k} g(Y, \nabla_X N_\ell)\, N_\ell \,, \end{aligned} \tag{1.25}$$

where N_ℓ is an orthonormal basis of vector fields for the normal space to Σ in a neighborhood of x.

The *mean curvature vector* H at x is by definition

$$H = \sum_{i=1}^{k} A(E_i, E_i) \,, \tag{1.26}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Furthermore, the *norm squared of the second fundamental form* at x is by definition

$$|A|^2 = \sum_{i,j=1}^{k} |A(E_i, E_j)|^2 \,. \tag{1.27}$$

Recall also that the *Gauss equations* assert that if X, $Y \in T_x\Sigma$, then

$$\begin{aligned} K_\Sigma(X,Y)\, |X \wedge Y|^2 &= K_M(X,Y)\, |X \wedge Y|^2 \\ &\quad + g(A(X,X),\, A(Y,Y)) - g(A(X,Y),\, A(X,Y)) \,, \end{aligned} \tag{1.28}$$

where $|X \wedge Y|^2$ is given by

$$|X \wedge Y|^2 = g(X,X)\, g(Y,Y) - g(X,Y)^2 \tag{1.29}$$

and $K_M(X,Y)$ and $K_\Sigma(X,Y)$ are the sectional curvatures of M and Σ, respectively, in the two-plane spanned by the vectors X and Y. If $\Sigma^{n-1} \subset$

M^n is a hypersurface and N is a unit normal vector field in a neighborhood of x, then

$$\nabla_{(\cdot)} N : T_x\Sigma \to T_x\Sigma \tag{1.30}$$

is a symmetric map (often referred to as the Weingarten map) and its eigenvalues $(\kappa_i)_{i=1,\dots,n-1}$ are called the principal curvatures. Moreover,

$$g(H, N) = -\sum_{i=1}^{n-1} \kappa_i \,. \tag{1.31}$$

Finally, if X is a vector field defined in a neighborhood of Σ, then the *divergence* of X at $x \in \Sigma$ is

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{n-1} g(\nabla_{E_i} X, E_i) \,, \tag{1.32}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Notice that $\operatorname{div}_\Sigma$ satisfies the Leibniz rule

$$\operatorname{div}_\Sigma(fX) = \langle \nabla_\Sigma f, X\rangle + f \operatorname{div}_\Sigma(X) \,. \tag{1.33}$$

We can also use $\operatorname{div}_\Sigma$ to define the Laplace operator Δ_Σ on Σ by

$$\Delta_\Sigma f = \operatorname{div}_\Sigma(\nabla_\Sigma f) \,. \tag{1.34}$$

A function f is said to be *harmonic* on Σ if $\Delta_\Sigma f = 0$.

Remark 1.3. *Note that*

$$\begin{aligned} \operatorname{div}_\Sigma Y^N &= \sum_i g(E_i, \nabla_{E_i} Y^N) = -\sum_i g(Y^N, \nabla_{E_i} E_i) \\ &= -g(Y^N, H) \,. \end{aligned} \tag{1.35}$$

1.3. The first variation formula. Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside a compact set,

$$F(x, 0) = x \,, \tag{1.36}$$

and for all $x \in \partial\Sigma$,

$$F(x, t) = x \,. \tag{1.37}$$

The vector field F_t restricted to Σ is often called the *variational vector field.* Now we want to compute the first variation of area for this one-parameter family of surfaces. Let x_i be local coordinates on Σ. Set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j}) \,, \tag{1.38}$$

$$\nu(t) = \sqrt{\det(g_{ij}(t))}\,\sqrt{\det(g^{ij}(0))} \,, \tag{1.39}$$

where a^{ij} denotes the inverse of the matrix a_{ij}. Note that $\nu(t)$ is well defined, independent of the choice of a coordinate system on Σ (since $\det(g_{ij}(t))$ changes by the determinant squared of the differential of a coordinate transformation while $\det(g^{ij}(0))$ changes by the inverse of this). Furthermore, the area formula is

$$\operatorname{Vol}(F(\Sigma,t)) = \int \nu(t)\,\sqrt{\det(g_{ij}(0))}\,; \tag{1.40}$$

where the integral is over Σ. Differentiating this gives

$$\frac{d}{dt}_{t=0}\operatorname{Vol}(F(\Sigma,t)) = \int \frac{d}{dt}_{t=0}\nu(t)\,\sqrt{\det(g_{ij}(0))}\,. \tag{1.41}$$

To evaluate $d/dt_{t=0}\nu(t)$ at some point x, we may choose the coordinate system such that at x it is orthonormal, i.e., so that at the point x,

$$g_{ij}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j\,, \\ 1 & \text{if } i = j\,. \end{cases} \tag{1.42}$$

Using this and the fact that the t and x_i derivatives commute (i.e., $\nabla_{F_t}F_{x_i} - \nabla_{F_{x_i}}F_t = [F_t, F_{x_i}] = 0$), we get at x,

$$\begin{aligned} \frac{d}{dt}_{t=0}\nu(t) &= \frac{1}{2}\sum_{i=1}^{k}\frac{d}{dt}\langle F_{x_i}, F_{x_i}\rangle = \sum_{i=1}^{k}\langle \nabla_{F_t}F_{x_i}, F_{x_i}\rangle \\ &= \sum_{i=1}^{k}\langle \nabla_{F_{x_i}}F_t, F_{x_i}\rangle = \operatorname{div}_\Sigma F_t\,. \end{aligned} \tag{1.43}$$

We can relate this formula to the mean curvature by writing the vector field F_t as the sum of its normal and tangential parts to get

$$\begin{aligned} \frac{d}{dt}_{t=0}\nu(t) &= \sum_{\ell=1}^{n-k}\sum_{i=1}^{k}\langle \nabla_{F_{x_i}}\langle F_t, N_\ell\rangle\, N_\ell, F_{x_i}\rangle + \operatorname{div}_\Sigma F_t^T \\ &= \sum_{\ell=1}^{n-k}\sum_{i=1}^{k}\langle F_t, N_\ell\rangle\,\langle \nabla_{F_{x_i}}N_\ell, F_{x_i}\rangle + \operatorname{div}_\Sigma F_t^T \\ &= -\langle F_t, H\rangle + \operatorname{div}_\Sigma F_t^T\,. \end{aligned} \tag{1.44}$$

Here N_ℓ is an orthonormal basis for the normal bundle of Σ at x. Integrating (1.43) and (1.44) gives the so-called first variation formula:

$$\frac{d}{dt}_{t=0}\operatorname{Vol}(F(\Sigma,t)) = -\int_\Sigma \langle F_t, H\rangle = \int_\Sigma \operatorname{div}_\Sigma F_t\,. \tag{1.45}$$

Note that Stokes' theorem was used to see that $\int \operatorname{div}_\Sigma F_t^T = 0$. As a consequence of (1.45), we see that Σ is a critical point for the area functional if and only if the mean curvature H vanishes identically.

Definition 1.4. (Minimal Submanifold) An immersed submanifold $\Sigma^k \subset M^n$ is said to be *minimal* if the mean curvature H vanishes identically.

It follows from the identity (1.45) that a graph in $\mathbb{R}^3$ is a minimal surface if and only if it satisfies the minimal surface equation (1.6).

2. Examples of Minimal Surfaces in $\mathbb{R}^3$

Minimal surfaces are characterized as having mean curvature zero. The simplest example is when the full second fundamental form (and not just its trace) vanishes, i.e., when the surface is totally geodesic. Since the second fundamental form is the derivative of the unit normal, this means that the normal is constant and, thus, the surface is a plane.

2.1. Topology of surfaces. Compact orientable surfaces without boundaries are classified by their genus, a nonnegative integer. Genus $= 0$ corresponds to a sphere, genus $= 1$ to the torus. A surface of genus $= k$ is modelled by the surface of a sphere to which k-handles have been attached. A compact orientable surface with a boundary is one formed by taking one of these surfaces and removing a number of disjoint disks. The genus of the surface with a boundary is the genus of the original object, and the boundary corresponds to the edges of the surface created by disk removal. In particular, a surface with genus 0 and nonempty boundary is a planar domain, i.e., it can be obtained from the disk in the plane by removing a number of disjoint sub-disks. Sometimes we will talk about surfaces that are simply connected. By this we will mean that every loop on the surface can be shrunk (without leaving the surface) to a point curve. One can easily see that the only simply connected surfaces are the disk and the sphere.

One immediate consequence of the definition of the genus is monotonicity under inclusion:

Lemma 1.5. *If Σ has genus k and $\Sigma_0 \subset \Sigma$, then the genus of Σ_0 is at most k.*

2.2. The helicoid; see Figure 1.1. The helicoid is given as the set $x_3 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$; alternatively, it is given in parametric form by

$$(x_1, x_2, x_3) = (t\,\cos s, t\,\sin s, s)\,, \tag{1.46}$$

where s, $t \in \mathbb{R}$. It was discovered by Meusnier (a student of Monge) in 1776. It is complete, embedded, singly-periodic and simply connected.

The helicoid is a ruled surface since its intersections with horizontal planes $\{x_3 = s\}$ are straight lines. These lines lift and rotate with constant speed to form a double spiral staircase. In 1842, Catalan showed that the

helicoid is the only (nonflat) ruled minimal surface; see Chapter 8 for further uniqueness results. A surface is said to be "ruled" if it can be parameterized by

$$X(s,t) = \beta(t) + s\,\delta(t) \text{ where } s,t \in \mathbb{R}, \tag{1.47}$$

and β and δ are curves in $\mathbb{R}^3$. The curve $\beta(t)$ is called the "directrix" of the surface, and a line having $\delta(t)$ as a direction vector is called a "ruling". For the standard helicoid, the x_3-axis is a directrix, and for each fixed t the line $s \to (s\cos t, s\,\sin t, t)$ is a ruling.

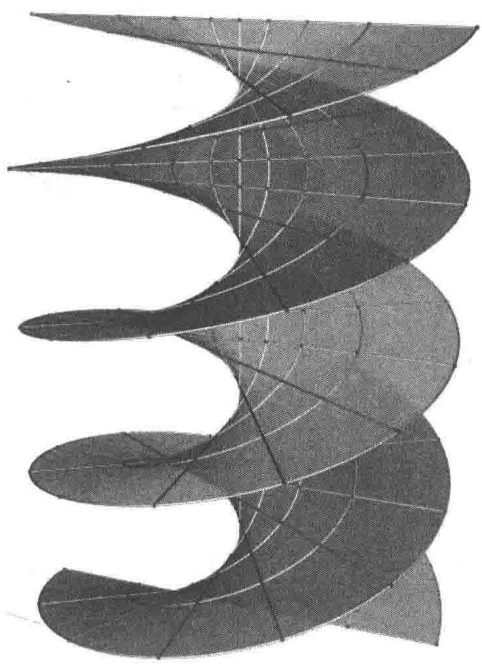

Figure 1.1. The helicoid, with the ruling pictured.
Credit: Matthias Weber, www.indiana.edu/~minimal.

2.3. The catenoid. The catenoid, shown in Figure 1.2, is the only nonflat minimal surface of revolution. It was discovered by Euler in 1744 and shown to be minimal by Meusnier (a student of Monge) in 1776. It is a complete embedded topological annulus (i.e., genus zero and two ends) and is given as the set where $x_1^2 + x_2^2 = \cosh^2(x_3)$ in $\mathbb{R}^3$. It is easy to see that the catenoid has finite total curvature.

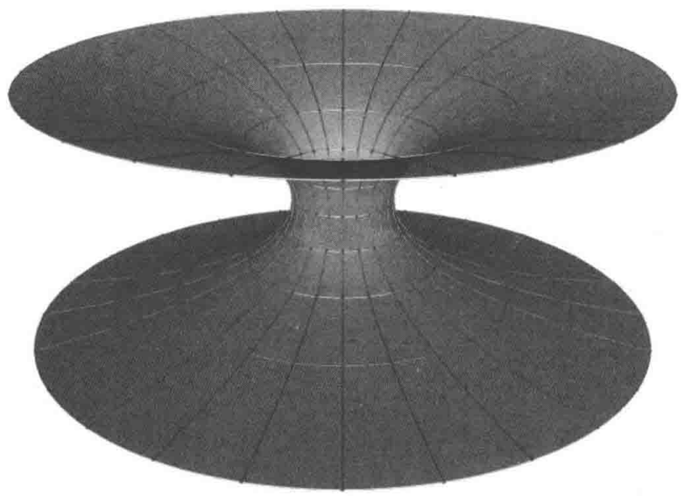

Figure 1.2. The catenoid given by revolving $x_1 = \cosh x_3$ around the x_3-axis. Credit: Matthias Weber, www.indiana.edu/~minimal.

2.4. Scherk's doubly-periodic surface. Approximately 70 years later and despite considerable work[1] by Monge, Legendre and Poisson, in 1835, Scherk, [**Sk**], discovered the next complete minimal surface.

Scherk's surface was a doubly-periodic minimal surface defined as a graph over the white squares of a checkerboard with vertical lines at the corners. A fundamental domain of Scherk's surface is given as a graph over the square $|x_1| < \pi/2$ and $|x_2| < \pi/2$ of

$$x_3 = \log \frac{\cos(x_2)}{\cos(x_1)}. \tag{1.48}$$

If we quotient out by the translations to get a minimal surface in $T^2 \times \mathbb{R}$, the resulting surface has genus 0 and 4 annular ends.

[1]See [**Os5**] for more on the history.

Let us check that Scherk's surface is, in fact, a minimal surface. We need only check that $x_3 = x_3(x_1, x_2)$ satisfies the minimal surface equation. Clearly,

$$\begin{aligned}
\partial_{x_1} x_3 &= \tan(x_1)\,, \\
\partial_{x_2} x_3 &= -\tan(x_2)\,, \\
\partial_{x_1 x_1} x_3 &= 1 + \tan^2(x_1)\,, \\
\partial_{x_2 x_2} x_3 &= -1 - \tan^2(x_2)\,, \\
\partial_{x_1 x_2} x_3 &= 0\,.
\end{aligned}$$

Hence, x_3 satisfies the minimal surface equation (1.7).

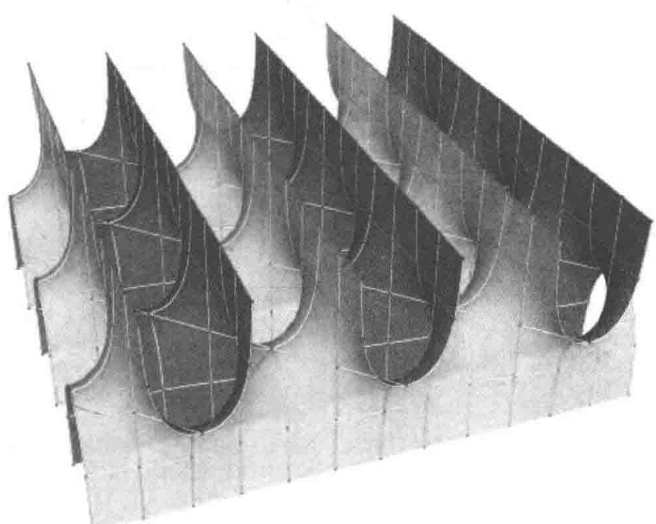

Figure 1.3. Scherk's doubly-periodic surface.
Credit: Matthias Weber, www.indiana.edu/~minimal.

2.5. Scherk's singly-periodic surface. Also in 1835, Scherk, [**Sk**], discovered a singly-periodic minimal surface that looks like the desingularization of a pair of orthogonal planes. This surface is still one of the most

important examples, as it plays a critical role in gluing and desingularization constructions for minimal and constant mean curvature surfaces; see Kapouleas, [**Ka**], and Traizet, [**Tr1**].

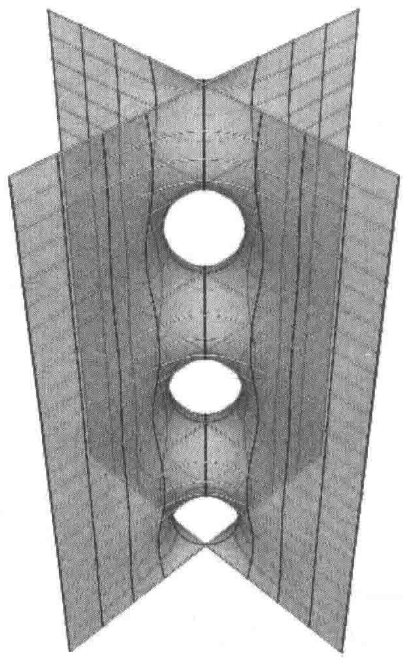

Figure 1.4. Scherk's singly-periodic surface.
Credit: Matthias Weber, www.indiana.edu/~minimal.

2.6. The Riemann Examples. Around 1860, Riemann, [**Ri**], classified all minimal surfaces in $\mathbb{R}^3$ that are foliated by circles and straight lines in horizontal planes. He showed that the only such surfaces are the plane, the catenoid, the helicoid, and a two-parameter family that is now known as the Riemann examples. The surfaces that he discovered formed a family of complete embedded minimal surfaces that are singly-periodic and have genus zero. Each of the surfaces has infinitely many parallel planar ends connected by necks ("pairs of pants").

Modulo rigid motions, this is a 2-parameter family of minimal surfaces. The parameters are:

- Neck size.
- Angle between period vector and the ends.

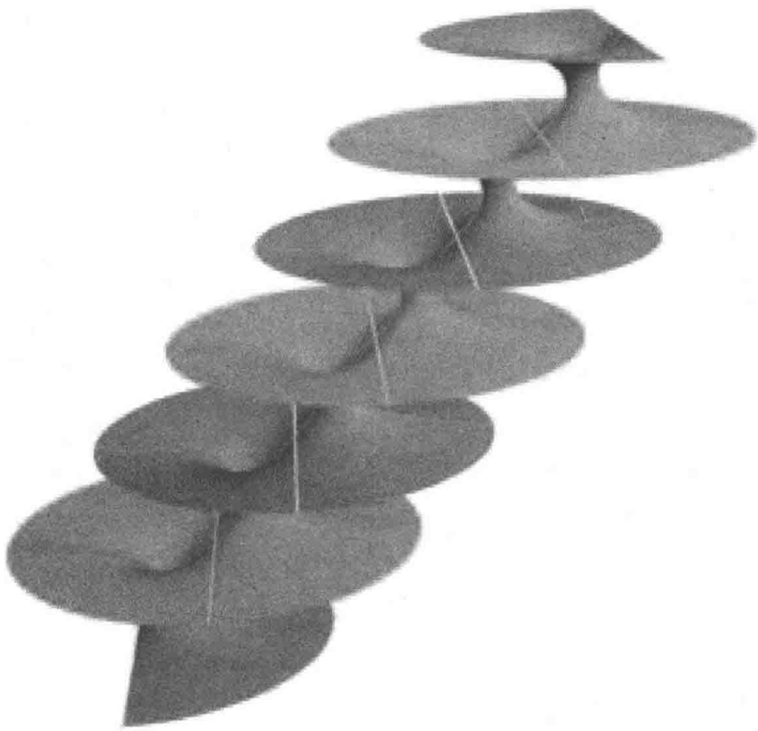

Figure 1.5. The Riemann examples.
Credit: Matthias Weber, www.indiana.edu/~minimal.

If we keep the neck size fixed and allow the angle to become vertical (i.e., perpendicular to the planar ends), the family degenerates to a pair of oppositely oriented helicoids. On the other hand, as the angle goes to zero, the family degenerates to a catenoid.

2.7. Enneper's surface. In 1864, Enneper, [**E**], discovered a minimal surface parameterized by

$$(x_1, x_2, x_3) = (s - s^3/3 + st^2, -t - s^2t + t^3/3, s^2 - t^2)\,, \tag{1.49}$$

where s, $t \in \mathbb{R}$. Unlike the earlier examples, Enneper's surface is not embedded. Like the catenoid, Enneper has finite total curvature.

2.8. The Costa-Hoffman-Meeks surfaces. In 1984, C. Costa, [**Cc**], discovered a new minimal surface with finite total curvature, genus one and three ends, each of which was embedded. This surface was constructed by the Weierstrass representation using clever choices of data on the torus. In

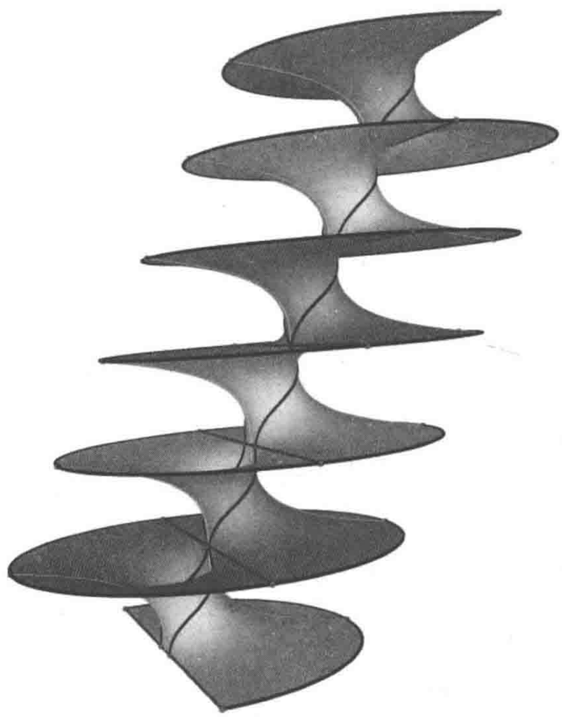

Figure 1.6. The Riemann examples: starting to degenerate to helicoids. Credit: Matthias Weber, www.indiana.edu/~minimal.

1985, D. Hoffman and W. Meeks, [**HoMe1**], proved that Costa's surface was embedded; this surface is now known as the Costa-Hoffman-Meeks surface.

Moreover, Hoffman and Meeks showed that Costa's surface was just the first in a family of surfaces with three ends and increasing genus. As the genus goes to infinity, the surfaces converge (at least as sets) to the union of a plane and a catenoid.

2.9. The Weber-Wolf surfaces. M. Weber and M. Wolf, [**WeWo2**], constructed a series of embedded minimal surfaces with two catenoidal ends (the top and bottom ends), any number of planar ends in the middle, and at least as many handles as there are planar ends. These surfaces are conjectured to be at the boundary of what is possible:

Conjecture 1.6 (Hoffman-Meeks). *If Σ is a complete embedded minimal surface in $\mathbb{R}^3$ with k ends and genus g, then $k \leq g + 2$.*

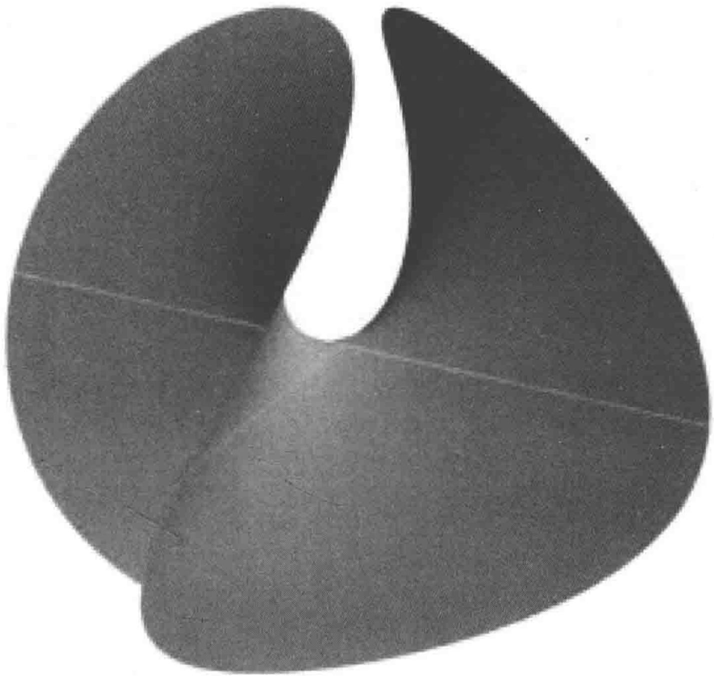

Figure 1.7. A small piece of Enneper's surface that is embedded; it extends to cross itself. Credit: Matthias Weber, www.indiana.edu/~minimal.

2.10. The genus one helicoid. In 1993, Hoffman-Karcher-Wei gave numerical evidence for the existence of a complete embedded minimal surface with genus one that is asymptotic to a helicoid; they called it a "genus one helicoid". In [**HoWW**], Hoffman, Weber and Wolf constructed such a surface as the limit of "singly-periodic genus one helicoids", where each singly-periodic genus one helicoid was constructed via the Weierstrass representation. Later, Hoffman and White constructed a genus one helicoid variationally in [**HoWh1**].

2.11. Building blocks for embedded surfaces. The Riemann examples, the genus one helicoid and the Costa-Hoffman-Meeks surface are all embedded minimal surfaces with finite genus and each can be thought of as being built out of pieces of planes, helicoids and catenoids. In Chapter 8, we will discuss the key Colding-Minicozzi structure result for embedded minimal surfaces. This result, from [**CM3**]–[**CM7**], proved that any embedded

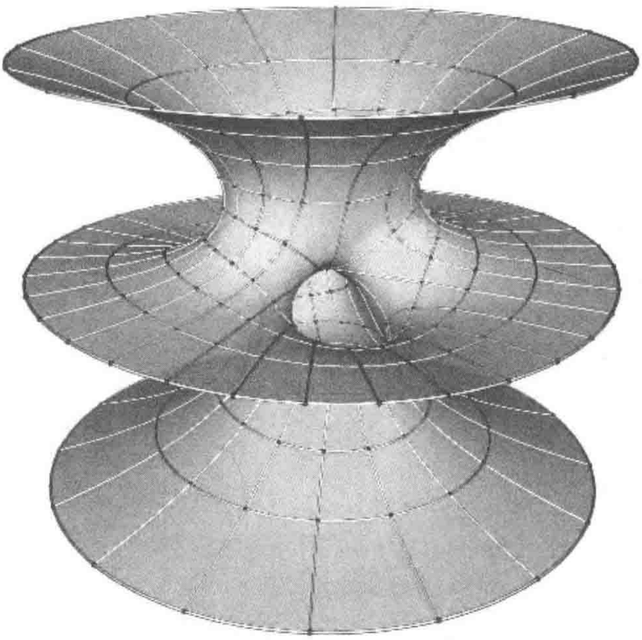

Figure 1.8. The original Costa-Hoffman-Meeks surface.
Credit: Matthias Weber, www.indiana.edu/~minimal,

minimal surface with finite genus can be built out of planes, helicoids and catenoids.

2.12. Schwarz's triply periodic minimal surfaces. In the late 19th century, H.A. Schwarz (of the Schwarz alternating method discussed in Chapter 5) and his students, including Neovius, [**Ne**], discovered five triply periodic minimal surfaces in $\mathbb{R}^3$; see [**Sz2**], [**Sz3**]. All of these surfaces have infinite genus, but their quotients by the crystallographic group have finite genus. In 1970, A. Schoen, [**Sca**], constructed many new families of triply periodic minimal surfaces, including the gyroid. These were rigorously shown to exist by Karcher in 1989, [**Kr**]. The gyroid was proven to be embedded by K. Grosse-Brauckmann and M. Wohlgemuth in 1996, [**GrW**]. Triply periodic minimal surfaces play a role in the sciences; see [**AHHT**]. The gyroid structure, for instance, is found in certain block copolymers.

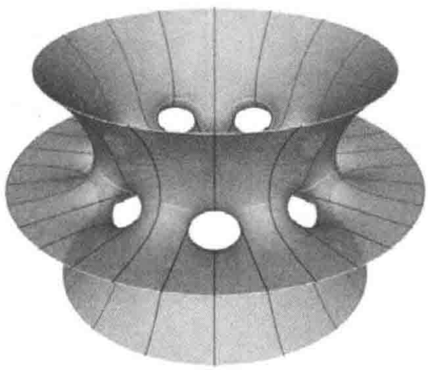

Figure 1.9. The Costa-Hoffman-Meeks surface with many handles. Credit: Matthias Weber, www.indiana.edu/~minimal.

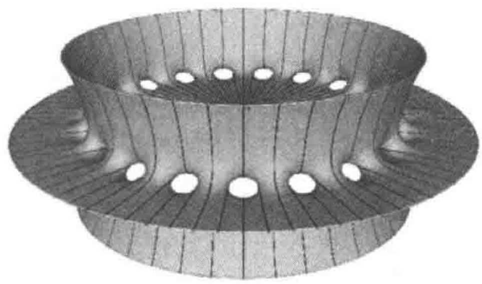

Figure 1.10. As the number of handles gets large, one can see a bent version of Scherk's singly-periodic surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

The fundamental domain for the Schwarz D-surface, discovered in 1865, solves the Plateau problem with boundary contained in the edges of a cube. The fundamental domain for the Schwarz P-surface, discovered in the 1880s, can also be constructed by solving a Plateau problem, this time with boundary equal to a 4-gon with corners at the vertices of a regular octahedron. This idea of solving a Plateau problem with a polygonal boundary and then using reflection has been used with great success, including H.B. Lawson's construction of minimal surfaces in the round 3-sphere, [**La1**]; cf. subsection 6.2.

2.13. The Chen-Gackstatter surface. In 1981, Chen and Gackstatter, [**CcG**], discovered the first complete immersed minimal surface of genus one with finite total curvature. The Chen-Gackstatter surface has one end and this is of Enneper type.

M. Weber and M. Wolf, [**WeWo1**], showed that one can add handles to the Chen-Gackstatter surface to get surfaces with arbitrarily high genus and one Enneper end. In her thesis, [**Ks**], S. Kim proved that the Weber-Wolf examples converge to Scherk's singly-periodic surface as the number of handles goes to infinity.

2.14. The higher dimensional catenoid. There is a higher dimensional analog of the catenoid given as a minimal surface of revolution around an axis. We refer to section 2 of [**TZ**] for its main properties. The most important ones from our point of view are:

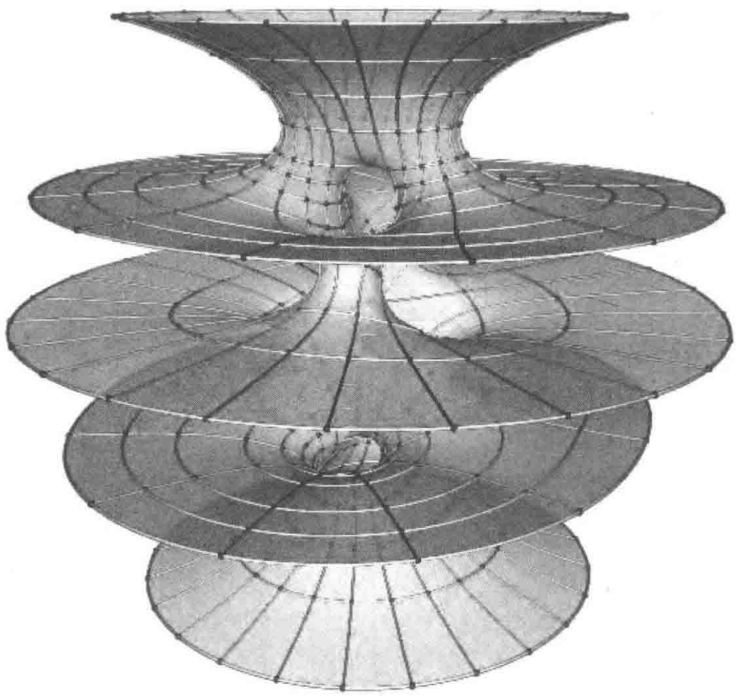

Figure 1.11. The Weber-Wolf surface of genus 4 with 5 ends.
Credit: Matthias Weber, www.indiana.edu/~minimal.

- It is a complete, embedded minimal hypersurface.
- The n-dimensional catenoid is given by revolving a curve around an axis and, thus, it is topologically $\mathbb{R} \times \mathbf{S}^{n-1}$. In particular, it is simply connected for $n > 2$.
- The volume of the n-dimensional catenoid in a ball of radius R is bounded by twice the volume of $B_R \subset \mathbb{R}^n$ and it converges to this as R goes to infinity.
- When $n > 3$, the catenoids lie in a slab between two parallel hyperplanes.

3. Consequences of the First Variation Formula

In this section, we will collect some important consequences of the first variation formula. The most important of these is the monotonicity formula,

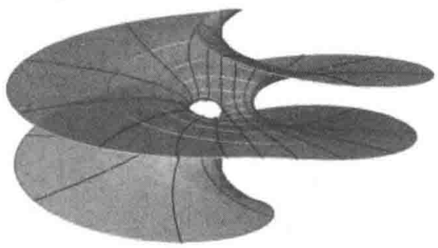

Figure 1.12. The genus one helicoid. Credit: Matthias Weber, www.indiana.edu/~minimal.

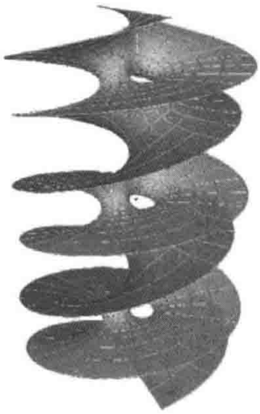

Figure 1.13. A periodic minimal surface asymptotic to the helicoid, whose fundamental domain has genus one. Credit: Matthias Weber, www.indiana.edu/~minimal.

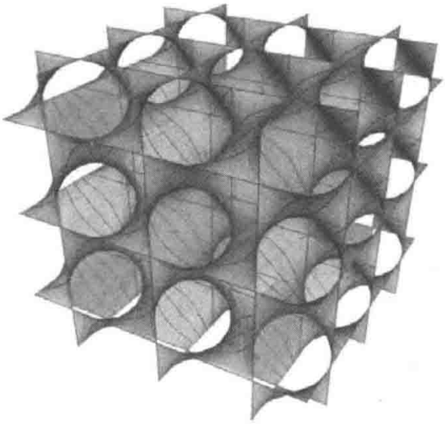

Figure 1.14. The Schwarz D-surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

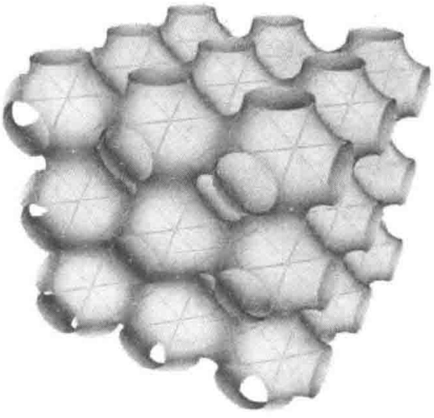

Figure 1.15. The Schwarz P-surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

Proposition 1.12. In later chapters, we will return to this subject. In Chapter 3, we extend these results to stationary varifolds, and in Chapter 5 to minimal surfaces in a three-manifold.

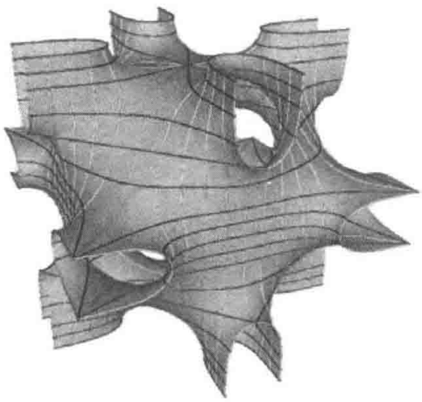

Figure 1.16. A fundamental domain of Neovius's minimal surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

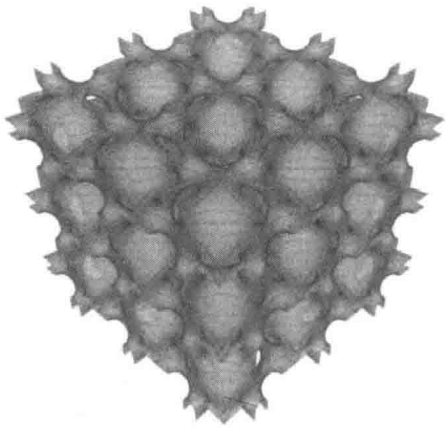

Figure 1.17. The triply periodic Neovius surface, discovered in 1883. Credit: Matthias Weber, www.indiana.edu/~minimal.

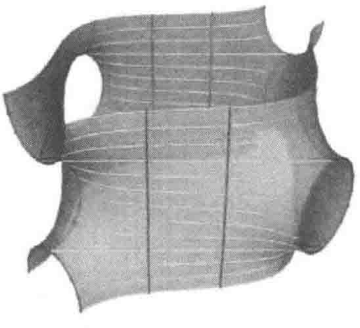

Figure 1.18. A fundamental domain of Schwarz's CLP surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

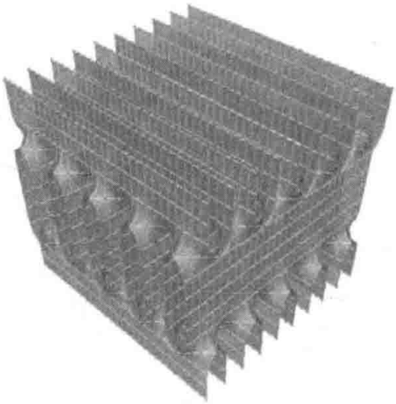

Figure 1.19. The triply periodic CLP surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

From (1.45), we see that Σ is minimal if and only if for all vector fields X with compact support and vanishing on the boundary of Σ,

$$\int_\Sigma \operatorname{div}_\Sigma X = 0\,. \tag{1.50}$$

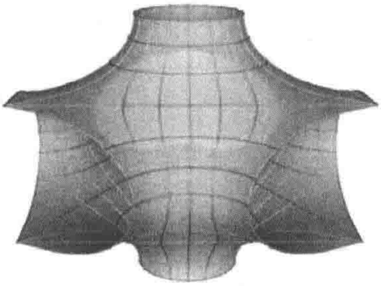

Figure 1.20. A fundamental domain of Schwarz's H-surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

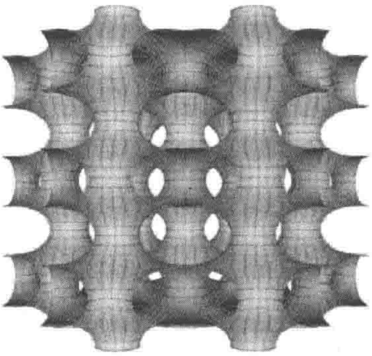

Figure 1.21. The triply periodic Schwarz H-surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

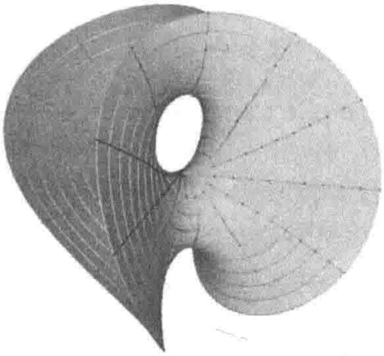

Figure 1.22. The Chen-Gackstatter surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

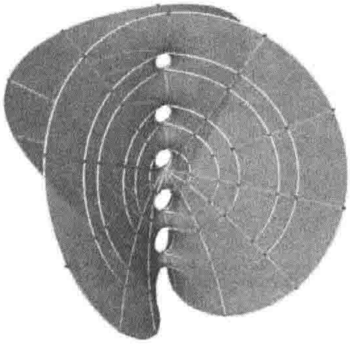

Figure 1.23. Adding handles to the Chen-Gackstatter surface. It looks more and more like Scherk's singly-periodic surface. Credit: Matthias Weber, www.indiana.edu/~minimal.

This equation is known as the first variation formula. It has the benefit that (1.50) makes sense as long as we can define the divergence on Σ. (This will later allow us to define a notion of "weak solution" for minimal surfaces.)

In the rest of this section, if $x_0 \in \mathbb{R}^n$ is fixed, then we let $B_s = B_s(x_0)$ be the Euclidean ball of radius s centered at x_0.

3.1. Harmonicity of the coordinate functions. As a consequence of (1.50), we will show the following proposition:

Proposition 1.7. *$\Sigma^k \subset \mathbb{R}^n$ is minimal if and only if the restrictions of the coordinate functions of $\mathbb{R}^n$ to Σ are harmonic functions.*

Proof. Let $e_i = \nabla_{\mathbb{R}^n} x_i$ be the i-th coordinate vector field on $\mathbb{R}^n$ and let η be a smooth function on Σ with compact support that vanishes on $\partial\Sigma$ (if Σ has boundary).

Since e_i is a constant vector field, the divergence of the compactly supported vector field $\eta\, e_i$ is given by

$$\mathrm{div}_\Sigma(\eta\, e_i) = \langle \nabla_\Sigma \eta \,, e_i \rangle = \langle \nabla_\Sigma \eta,\, \nabla_\Sigma x_i \rangle \,, \tag{1.51}$$

so we see that

$$-\int_\Sigma \langle \eta\, e_i, H \rangle = \int_\Sigma \mathrm{div}_\Sigma(\eta\, e_i) = \int_\Sigma \langle \nabla_\Sigma \eta,\, \nabla_\Sigma x_i \rangle = -\int_\Sigma \eta\, \Delta_\Sigma x_i \,. \tag{1.52}$$

From this, the claim follows easily. □

As a consequence, we can define a Flux homomorphism from $(k-1)$-dimensional homology classes to $\mathbb{R}^n$:

Corollary 1.8. *If $\Sigma^k \subset \mathbb{R}^n$ is minimal, then there is a well defined "Flux homomorphism" F from $(k-1)$-dimensional homology classes to $\mathbb{R}^n$ given by*

$$F_i([\gamma]) = \int_\gamma \langle n_\gamma, e_i \rangle \,, \tag{1.53}$$

where n_γ is the oriented conormal[2] to γ.

Proof. This mapping is clearly linear, so it suffices to show that it depends only on the homology class of γ. However, if γ_1 and γ_2 are homologous, then together they bound a k-chain Ω in Σ. Thus, minimality and the divergence theorem give

$$\int_{\gamma_1} \langle n_{\gamma_1}, e_i \rangle - \int_{\gamma_2} \langle n_{\gamma_2}, e_i \rangle = \int_\Omega \mathrm{div}_\Sigma\, e_i = 0 \,, \tag{1.54}$$

completing the proof. □

Another very useful consequence of (1.50) is a formula for the Laplacian of $|x|^2$ on a k-dimensional minimal surface Σ. Namely, since for any vector v we have

$$\nabla_v x_i = \langle v, e_i \rangle \,, \tag{1.55}$$

[2]The conormal is the vector that is tangent to Σ, but normal to γ.

we see that

$$\mathrm{div}_\Sigma(x_1, \ldots, x_n) = k \,. \tag{1.56}$$

Combining (1.56) and (1.50) (actually we just use that $\mathrm{div}_\Sigma Y^N = 0$ for any Y) gives

$$\Delta_\Sigma |x|^2 = 2\ \mathrm{div}_\Sigma(x_1, \ldots, x_n)^T = 2\ \mathrm{div}_\Sigma(x_1, \ldots, x_n) = 2k \,. \tag{1.57}$$

Recall that if $\Xi \subset \mathbb{R}^n$ is a compact subset, then the smallest convex set containing Ξ (the convex hull, $\mathrm{Conv}(\Xi)$) is the intersection of all half-spaces containing Ξ. In [**Os6**], Osserman showed that the maximum principle forces a minimal submanifold to lie in the convex hull of its boundary:

Proposition 1.9 (Convex Hull Property, [**Os6**]). *If $\Sigma^k \subset \mathbb{R}^n$ is a compact minimal submanifold, then $\Sigma \subset \mathrm{Conv}(\partial\Sigma)$.*

Proof. A half-space $H \subset \mathbb{R}^n$ can be written as

$$H = \{x \in \mathbb{R}^n \mid \langle x, e\rangle \leq a\} \,, \tag{1.58}$$

for a vector $e \in \mathbf{S}^{n-1}$ and constant $a \in \mathbb{R}$. By Proposition 1.7, the function $u(x) = \langle e, x\rangle$ is harmonic on Σ and hence attains its maximum on $\partial\Sigma$ by the maximum principle. □

The argument in the proof of the convex hull property can be rephrased to say that as we translate a hyperplane towards a minimal surface, the first point of contact must be on the boundary.

The convex hull property gives monotonicity of topology for minimal submanifolds. The simplest version of this is for minimal disks:

Corollary 1.10. *If $\Sigma^2 \subset \mathbb{R}^n$ is a compact minimal disk and K is a compact convex set with $K \cap \partial\Sigma = \emptyset$, then $K \cap \Sigma$ is simpy connected.*

Proof. Suppose instead that there is a closed curve $\gamma \subset K \cap \Sigma$ that does not bound a disk in $K \cap \Sigma$. Since Σ is simply connected, γ bounds a disk $\Gamma \subset \Sigma$. However, the minimal disk Γ now violates the convex hull property since $\partial\Gamma = \gamma$ lies inside K, but Γ does not. □

More generally, one gets monotonicity of the $(k-1)$-st homology group:

Lemma 1.11. *If $\Sigma^k \subset \mathbb{R}^n$ is a compact minimal submanifold and K is a compact convex set with $K \cap \partial\Sigma = \emptyset$, then the inclusion of $K \cap \Sigma$ into Σ is an injection on the $(k-1)$-st homology group.*

Proof. If not, then $B_R(0) \cap \Sigma$ contains a $(k-1)$-cycle γ that does not bound in $B_R(0) \cap \Sigma$ but does bound $\Gamma \subset \Sigma$. However, Γ then contradicts the convex hull property. □

3.2. Monotonicity. Before we state and prove the monotonicity formula of volume for minimal submanifolds, we will need to recall the coarea formula. This formula asserts (see, for instance, [**Fe**] or [**KP**] for a proof) that if Σ is a manifold and

$$h : \Sigma \to \mathbb{R}$$

is a proper (i.e., $h^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ, then for all locally integrable functions f on Σ and $t \in \mathbb{R}$,

$$\int_{\{h \leq t\}} f\,|\nabla_\Sigma h| = \int_{-\infty}^{t} \int_{h=\tau} f\, d\tau\,. \tag{1.59}$$

Proposition 1.12 (The Monotonicity Formula). *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$; then for all $0 < s < t$,*

$$\begin{aligned} &\frac{\mathrm{Vol}(B_t(x_0) \cap \Sigma)}{t^k} - \frac{\mathrm{Vol}(B_s(x_0) \cap \Sigma)}{s^k} \\ &\qquad = \int_{(B_t(x_0) \setminus B_s(x_0)) \cap \Sigma} \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}}\,. \end{aligned} \tag{1.60}$$

Proof. Define the function f on Σ by $f(x) = |x - x_0|$. Since Σ is minimal,

$$\Delta_\Sigma f^2 = 2\,\mathrm{div}_\Sigma(x - x_0) = 2k\,. \tag{1.61}$$

By Stokes' theorem integrating this gives

$$2\,k\,\mathrm{Vol}(\{f \leq s\}) = \int_{\{f \leq s\}} \Delta_\Sigma f^2 = 2 \int_{\{f = s\}} |(x - x_0)^T|\,. \tag{1.62}$$

The coarea formula (i.e., (1.59)) gives

$$\mathrm{Vol}(\{f \leq s\}) = \int_{\{f \leq s\}} |\nabla_\Sigma f|^{-1}\,|\nabla_\Sigma f| = \int_0^s \int_{f=\tau} |\nabla_\Sigma f|^{-1}\, d\tau\,. \tag{1.63}$$

Combining this with (1.62), an easy calculation gives

$$\begin{aligned} \frac{d}{ds}\left(s^{-k}\,\mathrm{Vol}(\{f \leq s\})\right) &= -k\,s^{-k-1}\,\mathrm{Vol}(\{f \leq s\}) + s^{-k} \int_{\{f=s\}} \frac{|x - x_0|}{|(x - x_0)^T|} \\ &= s^{-k-1} \int_{\{f=s\}} \left(\frac{|x - x_0|^2}{|(x - x_0)^T|} - |(x - x_0)^T|\right) \\ &= s^{-k-1} \int_{\{f=s\}} \frac{|(x - x_0)^N|^2}{|(x - x_0)^T|}\,. \end{aligned} \tag{1.64}$$

Integrating and applying the coarea formula once again gives the claim. □

Notice that $(x - x_0)^N$ vanishes precisely when Σ is conical about x_0, i.e., when Σ is invariant under dilations about x_0. As a corollary, we get the following:

Corollary 1.13. *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and $x_0 \in \mathbb{R}^n$; then the function*

$$\Theta_{x_0}(s) = \frac{\operatorname{Vol}(B_s(x_0) \cap \Sigma)}{\operatorname{Vol}(B_s \subset \mathbb{R}^k)} \tag{1.65}$$

is a nondecreasing function of s. Moreover, $\Theta_{x_0}(s)$ is constant in s if and only if Σ is conical about x_0.

Finally, if $x_0 \in \Sigma$, then $\Theta_{x_0}(s) \geq 1$; if for some $s > 0$, $\Theta_{x_0}(s) = 1$, then $B_s \cap \Sigma$ is a ball in some k-dimensional plane.

Proof. Proposition 1.12 directly shows that $\Theta_{x_0}(s)$ is monotone nondecreasing. Since Σ is smooth and proper, it is infinitesimally Euclidean and hence

$$\lim_{s \to 0} \Theta_{x_0}(s) \geq 1\,. \tag{1.66}$$

Combining monotonicity of $\Theta_{x_0}(s)$ with (1.66) shows that $\Theta_{x_0}(s) \geq 1$. If we have $\Theta_{x_0}(s) = 1$, then Θ_{x_0} is constant in s so that, by (1.60), $(x - x_0)^N$ is identically zero. Clearly, this implies that Σ is dilation invariant, and since Σ is smooth, Σ is contained in a k-plane. □

For later reference, we will record some consequences of Corollary 1.13. Let Σ be a minimal submanifold and define the *density* at x_0 by

$$\Theta_{x_0} = \lim_{s \to 0} \Theta_{x_0}(s)\,. \tag{1.67}$$

This limit, which exists since $\Theta_{x_0}(s)$ is monotone, is always at least 1 for $x_0 \in \Sigma$ by (1.66). In fact, so long as Σ is smooth, Θ_{x_0} is a nonnegative integer equal to the multiplicity of Σ at x_0. Note that if Σ is not embedded, then this multiplicity can be greater than one.

The next result, which is an elementary consequence of monotonicity, shows that this multiplicity is upper semicontinuous.

Corollary 1.14. *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, then the density Θ_x is an upper semicontinuous function on $\mathbb{R}^n$. Consequently, for any $\Lambda \geq 0$, the set*

$$\{x \in \Sigma \mid \Theta_x \geq \Lambda\} \tag{1.68}$$

is closed.

Proof. We need to show that if x_j is a sequence of points going to x, then

$$\Theta_x \geq \limsup_{x_j \to x} \Theta_{x_j}\,. \tag{1.69}$$

Given any $\delta > 0$, there exists an $s > 0$ such that

$$\Theta_x \geq \Theta_x(2s) - \delta\,, \tag{1.70}$$

and we can choose $0 < \epsilon < s$ so that

$$\Theta_x \geq (1 + s^{-1}\,\epsilon)^k\,\Theta_x(2s) - 2\,\delta\,. \tag{1.71}$$

For any x_j with $|x - x_j| < \epsilon$,

$$\begin{aligned}\Theta_{x_j} \leq \Theta_{x_j}(s) \leq \frac{\mathrm{Vol}(B_{s+\epsilon}(x) \cap \Sigma)}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)} &= (1 + s^{-1}\,\epsilon)^k\,\Theta_x(s+\epsilon) \\ \leq 2\delta + \Theta_x\,, &\end{aligned} \tag{1.72}$$

where the last inequality follows from (1.71). Since δ was arbitrarily small, (1.72) implies (1.69) and hence Θ is upper semicontinuous. It follows immediately that the set defined in (1.68) must be closed. □

3.3. The mean value inequality.

Proposition 1.15 (The Mean Value Inequality). *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and f is a function on Σ, then*

$$\begin{aligned} & t^{-k} \int_{B_t \cap \Sigma} f - s^{-k} \int_{B_s \cap \Sigma} f \\ &= \int_{(B_t \setminus B_s) \cap \Sigma} f \, \frac{|x^N|^2}{|x|^{k+2}} + \frac{1}{2} \int_s^t \tau^{-k-1} \int_{B_\tau \cap \Sigma} (\tau^2 - |x|^2)\, \Delta_\Sigma f \, d\tau\,. \end{aligned} \tag{1.73}$$

Proof. Observe that the monotonicity formula will be the special case where $f = 1$. Since Σ is minimal, integration by parts gives

$$\begin{aligned} 2k \int_{B_s \cap \Sigma} f &= \int_{B_s \cap \Sigma} f\, \Delta_\Sigma |x|^2 \\ &= \int_{B_s \cap \Sigma} |x|^2\, \Delta_\Sigma f + 2 \int_{\partial B_s \cap \Sigma} f\, |x^T| - s^2 \int_{B_s \cap \Sigma} \Delta_\Sigma f\,. \end{aligned} \tag{1.74}$$

Using this and the coarea formula (i.e., (1.59)) gives

$$\begin{aligned} & \frac{d}{ds} \left(s^{-k} \int_{B_s \cap \Sigma} f \right) \\ &= -k\, s^{-k-1} \int_{B_s \cap \Sigma} f + s^{-k} \int_{\partial B_s \cap \Sigma} f\, \frac{|x|}{|x^T|} \\ &= s^{-k-1} \int_{\partial B_s \cap \Sigma} f\, \frac{|x^N|^2}{|x^T|} + \frac{1}{2}\, s^{-k-1} \int_{B_s \cap \Sigma} (s^2 - |x|^2)\, \Delta_\Sigma f\,. \end{aligned} \tag{1.75}$$

Integrating and using the coarea formula gives the claim. □

For future reference, we next record a general mean value inequality which follows from Proposition 1.15.

Corollary 1.16. *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \Sigma$, and $s > 0$ satisfy $B_s(x_0) \cap \partial\Sigma = \emptyset$. If f is a nonnegative function on Σ with $\Delta_\Sigma f \geq -\lambda\, s^{-2} f$, then*

$$f(x_0) \leq e^{\frac{\lambda}{2}} \frac{\int_{B_s(x_0)\cap\Sigma} f}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)}. \tag{1.76}$$

Proof. If we define $g(t)$ by

$$g(t) = t^{-k} \int_{B_t(x_0)\cap\Sigma} f, \tag{1.77}$$

then Proposition 1.15 implies that

$$g'(t) \geq -\frac{\lambda}{2}\, s^{-2}\, t^{1-k} \int_{B_t(x_0)\cap\Sigma} f = -\frac{\lambda}{2}\, s^{-2}\, t\, g(t). \tag{1.78}$$

We can rewrite (1.78) as

$$\frac{g'(t)}{g(t)} \geq -\frac{\lambda}{2}\, s^{-2}\, t \geq -\frac{\lambda}{2s}. \tag{1.79}$$

From (1.79), it is obvious that $e^{\lambda t/(2s)}\, g(t)$ is monotone nondecreasing and (1.76) follows immediately. □

A function f is said to be subharmonic on Σ if $\Delta_\Sigma f \geq 0$. We get immediately the following mean value inequality for the special case of nonnegative subharmonic functions:

Corollary 1.17. *Suppose that $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \mathbb{R}^n$, and f is a nonnegative subharmonic function on Σ; then*

$$s^{-k} \int_{B_s(x_0)\cap\Sigma} f \tag{1.80}$$

is a nondecreasing function of s. In particular, if $x_0 \in \Sigma$, then for all $s > 0$,

$$f(x_0) \leq \frac{\int_{B_s(x_0)\cap\Sigma} f}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)}. \tag{1.81}$$

In Chapter 3, we will need a general mean value inequality when Σ is not minimal. We record this next:

Lemma 1.18. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a hypersurface with mean curvature H and f is a nonnegative function on Σ, then for $s < t$,*

$$t^{-k} \int_{B_t\cap\Sigma} f - s^{-k} \int_{B_s\cap\Sigma} f \geq \int_s^t \tau^{-n-1} \int_{B_\tau\cap\Sigma} \langle x, \nabla f + f\, H\, N\rangle\, d\tau. \tag{1.82}$$

Proof. The proof follows as in Proposition 1.15, except that we keep track of the terms involving H and throw away the nonnegative (and thus only helpful) terms involving x^N. We leave the details as an exercise. □

4. The Gauss Map

Let $\Sigma^2 \subset \mathbb{R}^3$ be a surface. The *Gauss map* is a continuous choice of a unit normal

$$N : \Sigma \to \mathbf{S}^2 \subset \mathbb{R}^3 . \tag{1.83}$$

Observe that there are two choices of such a map N and $-N$ corresponding to a choice of orientation of Σ. Moreover, the differential of the map N can be identified with the Weingarten map defined above. To see this, suppose that E_1, E_2 is an orthonormal frame on Σ. Since the unit normal to $\mathbf{S}^2$ at $N(x)$ is just $N(x)$ itself, E_1, E_2 also gives an orthonormal frame on the image. Using this identification, the differential dN is given by

$$\langle dN(E_i), E_j \rangle = \langle \nabla_{E_i} N, E_j \rangle = -\langle N, \nabla_{E_i} E_j \rangle = -A_{ij} . \tag{1.84}$$

In the last equality, we identified the normal vector A_{ij} with its inner product with N (since Σ is a hypersurface).

If Σ is minimal, then the Gauss map is an (anti) conformal map since the eigenvalues of the Weingarten map are κ_1 and $\kappa_2 = -\kappa_1$. Moreover, for a minimal surface

$$|dN|^2 = |A|^2 = \kappa_1^2 + \kappa_2^2 = -2\,\kappa_1\,\kappa_2 = -2\,K = -2\det(dN) , \tag{1.85}$$

and the area of the Gauss map is a multiple of the total curvature. This conformality of the Gauss map for a minimal surface in $\mathbb{R}^3$, namely (1.85), can be used to prove the classical Bernstein theorem described in the next section.

4.1. Local coordinates on a graph. We conclude this section by calculating the metric, curvature, and second fundamental form of a graph. If $\Sigma \subset \mathbb{R}^3$ is the graph of a function $u = u(x, y)$, then, as we have already seen, we can take

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} . \tag{1.86}$$

Using (x, y) as coordinates on the graph, we may express the induced metric g as

$$g_{xx} = (1 + u_x^2) , \quad g_{xy} = g_{yx} = u_x u_y , \quad g_{yy} = (1 + u_y^2) . \tag{1.87}$$

By direct calculation, the eigenvalues of the matrix g are 1 and $(1 + |\nabla u|^2)$. This can also easily be seen geometrically. Similarly, the inverse matrix is given by

$$g^{xx} = \frac{1 + u_y^2}{1 + |\nabla u|^2} , \quad g^{xy} = g^{yx} = \frac{-u_x\, u_y}{1 + |\nabla u|^2} , \quad g^{yy} = \frac{1 + u_x^2}{1 + |\nabla u|^2} . \tag{1.88}$$

By the Gauss equation, the Gauss curvature of the graph of u is given by

$$
\begin{aligned}
K = & \kappa_1 \kappa_2 = \frac{\langle N_x, (1,0,u_x)\rangle \langle N_y, (0,1,u_y)\rangle - \langle N_x, (0,1,u_y)\rangle^2}{|(1,0,u_x) \times (0,1,u_y)|^2} \\
= & \frac{\langle(-u_{xx}, -u_{yx}, 0), (1,0,u_x)\rangle \langle(-u_{xy}, -u_{yy}, 0), (0,1,u_y)\rangle}{(1+|\nabla u|^2)^2} \\
& - \frac{\langle(-u_{xx}, -u_{xy}, 0), (0,1,u_y)\rangle^2}{(1+|\nabla u|^2)^2} = \frac{u_{xx}\, u_{yy} - u_{xy}^2}{(1+|\nabla u|^2)^2} .
\end{aligned} \tag{1.89}
$$

Therefore,

$$
K \, d\,\mathrm{Area} = \frac{u_{xx}\, u_{yy} - u_{xy}^2}{(1+|\nabla u|^2)^{\frac{3}{2}}} \, dx \wedge dy . \tag{1.90}
$$

Similarly, we may express the second fundamental form A in the coordinates (x, y) as

$$
A_{xx} = \langle \nabla_{(1,0,u_x)}(1,0,u_x), N\rangle = \langle (0,0,u_{xx}), N\rangle = \frac{u_{xx}}{(1+|\nabla u|^2)^{\frac{1}{2}}} . \tag{1.91}
$$

Thus, this (and similar calculations) give

$$
\begin{aligned}
A_{xx} &= \frac{u_{xx}}{(1+|\nabla u|^2)^{\frac{1}{2}}} , \\
A_{xy} = A_{yx} &= \frac{u_{xy}}{(1+|\nabla u|^2)^{\frac{1}{2}}} , \\
A_{yy} &= \frac{u_{yy}}{(1+|\nabla u|^2)^{\frac{1}{2}}} .
\end{aligned} \tag{1.92}
$$

Recall that (by (1.27)) the norm squared of A is the sum of the squares of the entries in an orthonormal frame; in a general frame, this is given by

$$
|A|^2 = A_{ij}\, A_{kl}\, g^{ik}\, g^{jl} . \tag{1.93}
$$

The expression for the second fundamental form and the bound on the eigenvalues of g (see (1.87)) together imply

$$
\frac{|\,\mathrm{Hess}_u\,|^2}{(1+|\nabla u|^2)^3} \leq |A|^2 \leq 2 \frac{|\,\mathrm{Hess}_u\,|^2}{1+|\nabla u|^2} . \tag{1.94}
$$

5. The Theorem of Bernstein

Before we prove the famous theorem of Bernstein, we will give a bound for the total curvature of a minimal graph. We will later see that, with some more work, this bound can be used to give local curvature estimates for minimal graphs. Such a local curvature estimate was proven originally by Heinz [**He**] using complex analysis and provided a generalization of the theorem of Bernstein.

Lemma 1.19. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all nonnegative Lipschitz functions η with support contained in $\Omega \times \mathbb{R}$,*

$$\int_{\mathrm{Graph}_u} |A|^2 \eta^2 \leq C \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u} \eta|^2 . \tag{1.95}$$

Proof. Let ω denote the area two-form on the unit sphere $\mathbf{S}^2$. Since the upper hemisphere is contractible, we can find a (smooth) one-form α on the upper hemisphere so that

$$d\alpha = \omega . \tag{1.96}$$

As before, let N denote the Gauss map. Since Σ is minimal and the differential d commutes with pull-backs, we see that

$$|A|^2 \, d\,\mathrm{Area} = -2K \, d\,\mathrm{Area} = 2\, N^*\omega = 2\, dN^*\alpha . \tag{1.97}$$

Moreover, since α is a one-form, there is a constant C_α so that

$$|N^*\alpha| \leq C_\alpha \, |dN| = C_\alpha \, |A| . \tag{1.98}$$

Set $\Sigma = \mathrm{Graph}_u$. By (1.97), Stokes' theorem, and (1.98), we get

$$\begin{aligned} \int_\Sigma \eta^2 \, |A|^2 \, d\,\mathrm{Area} &= 2 \int_\Sigma \eta^2 \, dN^*\alpha = -4 \int_\Sigma \eta \, d\eta \wedge N^*\alpha \\ &\leq 4\, C_\alpha \int_\Sigma \eta \, |\nabla_\Sigma \eta| \, |A| \, d\,\mathrm{Area} \\ &\leq 4\, C_\alpha \left(\int_\Sigma \eta^2 \, |A|^2 \, d\,\mathrm{Area} \right)^{\frac{1}{2}} \left(\int_\Sigma |\nabla_\Sigma \eta|^2 \, d\,\mathrm{Area} \right)^{\frac{1}{2}} , \end{aligned} \tag{1.99}$$

where the last inequality used the Cauchy-Schwarz inequality. Finally, squaring this and dividing both sides by $\int_\Sigma \eta^2 \, |A|^2$ gives

$$\int_\Sigma \eta^2 \, |A|^2 \leq 16\, C_\alpha^2 \int_\Sigma |\nabla_\Sigma \eta|^2 . \tag{1.100}$$

This proves the lemma. □

5.1. The logarithmic cutoff trick. The previous lemma shows that we can bound the total curvature in terms of the energy of a cutoff function η. When the graph Σ is entire (i.e., defined over all of $\mathbb{R}^2$), we will see that it is possible to choose a sequence of η's that converge to one but has energy going to zero. This is possible when Σ has quadratic area growth and is known as the "logarithmic cutoff trick".

To see this, fix a large integer N and define η on $B_{e^{2N}} \subset \mathbb{R}^2$ by

$$\eta = \begin{cases} 1 & \text{if } r \leq e^N\,, \\ 2 - \log(r)/N & \text{if } e^N < r \leq e^{2N}\,, \\ 0 & \text{if } e^{2N} < r\,, \end{cases} \tag{1.101}$$

where $r = |x|$. Since $|\nabla \eta| = \frac{1}{N\,r}$ is radial, the energy of η is

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \eta|^2 &= 2\,\pi \int_{e^N}^{e^{2N}} \frac{1}{N^2\,r^2}\,r\,dr = \frac{2\,\pi}{N^2} \int_{e^N}^{e^{2N}} \frac{dr}{r} \\ &= \frac{2\,\pi}{N^2}\left[\log(e^{2N}) - \log(e^N)\right] = \frac{2\,\pi}{N}\,. \end{aligned} \tag{1.102}$$

In particular, by taking a sequence of N's going to infinity, we get a sequence of cutoffs that converges to one and has energy going to 0. Clearly, the same argument applies with r equal to the distance to a fixed point on a manifold Σ satisfying

$$\mathrm{Vol}(\partial B_r) \leq C\,r$$

for any fixed constant C. However, the argument can be pushed through when Σ has quadratic area growth

$$\mathrm{Vol}(B_r) \leq C\,r^2$$

by breaking the integral up into integrals over annuli with bounded ratio between the inner and outer circles. Namely, using that

$$\sup_{B_{e^\ell}\setminus B_{e^{\ell-1}}} |\nabla \eta|^2 \leq N^{-2}\,e^{2-2\ell}\,,$$

we get

$$\begin{aligned} \int |\nabla \eta|^2 &\leq \sum_{\ell=N+1}^{2N} \int_{B_{e^\ell}\setminus B_{e^{\ell-1}}} \left[N^{-2}\,e^{2-2\ell}\,\mathrm{Vol}(B_{e^\ell})\right] \\ &\leq C\,e^2\,N^{-2} \sum_{\ell=N+1}^{2N} \int_{B_{e^\ell}\setminus B_{e^{\ell-1}}} = \frac{C\,e^2}{N}\,. \end{aligned} \tag{1.103}$$

We next combine the previous lemma and the logarithmic cutoff trick to get a total curvature bound for a minimal graph.

Corollary 1.20. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$, and Ω contains a ball of radius $\kappa\,R$ centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}\,R}\cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log \kappa}\,. \tag{1.104}$$

Proof. Set $\Sigma = \mathrm{Graph}_u$. Define the cutoff function η on all of $\mathbb{R}^3$ and then restrict it to the graph of u as follows: Let r denote the distance to the origin in $\mathbb{R}^3$ and define η by

$$\eta = \begin{cases} 1 & \text{if } r^2 \le \kappa\, R^2\,, \\ 2 - 2\,\log(r\,R^{-1})/\log\kappa & \text{if } \kappa\, R^2 < r^2 \le \kappa^2\, R^2\,, \\ 0 & \text{if } r^2 > \kappa^2\, R^2\,. \end{cases} \tag{1.105}$$

Since $|\nabla_\Sigma r| \le |\nabla r| = 1$, we have

$$|\nabla_\Sigma \eta| \le \frac{2}{r\,\log\kappa}\,. \tag{1.106}$$

Applying Lemma 1.19 with this cutoff function η and using the area bound (1.20), we get

$$\begin{aligned} \int_{B_{\sqrt{\kappa}\,R}\cap\Sigma} |A|^2 &\le \int_\Sigma \eta^2\,|A|^2 \le C\int_\Sigma |\nabla_\Sigma\eta|^2 \le \frac{4\,C}{(\log\kappa)^2}\int_{B_{\kappa\,R}\cap\Sigma} r^{-2} \\ &\le \frac{4\,C}{(\log\kappa)^2}\sum_{\ell=(\log\kappa)/2}^{\log\kappa}\int_{(B_{\mathrm{e}^\ell\,R}\setminus B_{\mathrm{e}^{\ell-1}\,R})\cap\Sigma} r^{-2} \\ &\le \frac{4\,C}{(\log\kappa)^2}\sum_{\ell=(\log\kappa)/2}^{\log\kappa} 2\,\pi\,\mathrm{e}^2 \le \frac{4\,\pi\,C\,\mathrm{e}^2}{\log\kappa}\,. \end{aligned} \tag{1.107}$$

(Here, for simplicity, we assumed that $(\log\kappa)/2$ is an integer.) □

As mentioned above, this argument, i.e., integration by parts with this particular choice of η, is often referred to as "a logarithmic cutoff argument". It is quite useful when the surface has at most quadratic area growth.

As a consequence of this corollary, we get the following theorem of S. Bernstein [**Be**] from 1916:

Theorem 1.21 (Bernstein, [**Be**]). *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then $u(x,y) = ax + by + c$ for some constants a, b, $c \in \mathbb{R}$.*

Proof. By the previous corollary, we have for all $R > 1$,

$$\int_{B_{\sqrt{R}}\cap\mathrm{Graph}_u} |A|^2 \le \frac{C}{\log R}\,. \tag{1.108}$$

Letting $R \to \infty$, we conclude that $|A|^2 \equiv 0$; hence $0 = u_{xx} = u_{xy} = u_{yy}$ and therefore $u = ax + by + c$ for some constants a, b, $c \in \mathbb{R}$. □

The previous proof (due to L. Simon [**Si5**]) of the theorem of Bernstein relied on minimality for two facts:

- The area bound for minimal graphs, (1.20).
- The conformality of the Gauss map, (1.85).

This proof can actually be applied to a wider class of differential equations where the conformality is replaced by quasi-conformality. We will briefly return to this later (in (7.7), where we also define quasi-conformality), but we will not discuss estimates for quasi-conformal maps in these notes. A detailed discussion may be found in Chapter 16 of [**GiTr**].

Furthermore, the argument of Lemma 1.19 applies any time that the Gauss map omits an open set in $\mathbf{S}^2$. Using this, it is then not hard to extend the Bernstein theorem to complete minimal surfaces whose Gauss maps omit an open set; this was a conjecture of L. Nirenberg and was proven by R. Osserman (by a different method), [**Os4**].

6. The Weierstrass Representation

The classical Weierstrass representation takes holomorphic data (a Riemann surface, a meromorphic function, and a holomorphic one-form) and associates a minimal surface in $\mathbb{R}^3$. To be precise, given

- a Riemann surface Ω,
- a meromorphic function g on Ω,
- a holomorphic one-form ϕ on Ω,

then we get a (branched) conformal minimal immersion $F : \Omega \to \mathbb{R}^3$ by
(1.109)
$$F(z) = \text{Re} \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2} (g^{-1}(\zeta) - g(\zeta)), \frac{i}{2} (g^{-1}(\zeta) + g(\zeta)), 1 \right) \phi(\zeta) \, .$$

Here $z_0 \in \Omega$ is a fixed base point and the integration is along a path $\gamma_{z_0,z}$ from z_0 to z. The choice of z_0 changes F by adding a constant. In general, the map F may depend on the choice of path (and hence may not be well defined); this is known as "the period problem" (see M. Weber and M. Wolf, [**WeWo2**], for the latest developments).

Lemma 1.22. *If f^1, f^2, f^3 are holomorphic functions on $\Omega \subset \mathbf{C}$ and $F = (F^1, F^2, F^3) : \Omega \to \mathbb{R}^3$ is given by*

$$F(z) = Re \int_{\zeta \in \gamma_{z_0,z}} (f^1, f^2, f^3) \, d\zeta \, , \tag{1.110}$$

then for each $i = 1, 2, 3$ we get

$$\frac{\partial F^i}{\partial x} - i \, \frac{\partial F^i}{\partial y} = f^i \, . \tag{1.111}$$

Proof. If $g = u + i\,v$ is holomorphic, then the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \tag{1.112}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,. \tag{1.113}$$

In particular, we get

$$\begin{aligned} 2\,g' &= \frac{\partial g}{\partial x} - i\,\frac{\partial g}{\partial y} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ &= 2\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right)\,. \end{aligned} \tag{1.114}$$

□

As an immediate consequence of Lemma 1.22, we see that the Weierstrass representation gives a conformal mapping (see (1.116) below). Since harmonic functions are conformally invariant in dimension two, it follows that the restrictions of the coordinate functions to the image of a Weierstrass representation are harmonic. Consequently, we see that the image is in fact a minimal surface.

The unit normal N, metric ds^2, and Gauss curvature K of the resulting surface are then

$$N = \frac{\left(2\,\mathrm{Re}\ g, 2\,\mathrm{Im}\ g, |g|^2 - 1\right)}{|g|^2 + 1}\,, \tag{1.115}$$

$$ds^2 = \frac{|\phi|^2}{4}\,\left(|g|^{-1} + |g|\right)^2\,, \tag{1.116}$$

$$\mathrm{K} = -\left[\frac{4|\partial_z g|\,|g|}{|\phi|\,(1 + |g|^2)^2}\right]^2\,. \tag{1.117}$$

Since the pullback $F^*(dx_3)$ is Re ϕ by (1.109), ϕ is usually called the *height differential.* By (1.115), g is the composition of the Gauss map followed by stereographic projection.

The standard constructions of minimal surfaces from Weierstrass data are

(1.118) $g(z) = z$, $\phi(z) = dz/z$, $\Omega = \mathbf{C} \setminus \{0\}$ giving a catenoid,

(1.119) $g(z) = \mathrm{e}^{iz}$, $\phi(z) = dz$, $\Omega = \mathbf{C}$ giving a helicoid,

(1.120) $g(z) = 1/z$, $\phi(z) = 2\,z\,dz$, $\Omega = \mathbf{C}$ giving Enneper's surface.

The representation (1.109) gives an obvious way to deform a minimal immersion. Namely, multiplying the one-form ϕ by a unit complex number $\mathrm{e}^{i\theta}$ gives another minimal immersion. Moreover, since the metric ds^2 only depends on $|\phi|$, this new surface is isometric to the original one. When

$\theta = \pi/2$, the new surface is called the *conjugate* minimal surface; for general values of θ, these are called associate minimal surfaces. Using a change of variables, it is not hard to see that the helicoid and the (universal cover of the) catenoid are conjugate minimal surfaces.

The Weierstrass representation is particularly useful for constructing immersed minimal surfaces. For example, in [**Na1**], Nadirashvili used it to construct a complete immersed minimal surface in the unit ball in $\mathbb{R}^3$. In particular, Nadirashvili's surface is not proper, i.e., the intersections with compact sets are not necessarily compact.

Typically, it is rather difficult to prove that the resulting immersion is an embedding (i.e., is 1–1), although there are some interesting cases where this can be done. The first modern example was [**HoMe1**] where D. Hoffman and W. Meeks proved that the surface constructed by Costa was embedded; this was the first new complete finite topology properly embedded minimal surface discovered since the classical catenoid, helicoid, and plane. This led to the discovery of many more such surfaces (see [**HoK**], [**Os2**], and [**Ro**] for more discussion).

6.1. Finite total curvature. A surface $\Sigma^2 \subset \mathbb{R}^n$ is said to have *finite total curvature* if

$$\int_\Sigma |A|^2 < \infty \, .$$

When $\Sigma \subset \mathbb{R}^3$ is minimal, the Gauss equation implies that this is equivalent to

$$\int_\Sigma |K_\Sigma| = - \int_\Sigma K_\Sigma < \infty \, .$$

In 1964, R. Osserman, [**Os3**], showed that a complete minimal surface in $\mathbb{R}^3$ is conformally equivalent to a closed Riemann surface with a finite number of points removed and the Gauss map extends meromorphically across the punctures.

6.2. The Schwarz reflection principle. The Schwarz reflection principle gives a way to extend solutions of a differential equation past the boundary, assuming that the solution has a nice form along the boundary. We will see versions of this for harmonic functions, holomorphic functions, and minimal surfaces.

We begin with the reflection principle for harmonic functions. The first observation is that harmonic functions inherit the symmetries of their boundary data:

Lemma 1.23. *Suppose that $v : D \to \mathbb{R}$ is harmonic and continuous on $\overline{D}$.*

(1) *If $v(x, y) = v(x, -y)$ on ∂D, then $v(x, y) = v(x, -y)$ on $\overline{D}$.*

(2) *If* $v(x,y) = -v(x,-y)$ *on* ∂D, *then* $v(x,y) = -v(x,-y)$ *on* $\overline{D}$.

Proof. Define a reflected function $w : D \to \mathbb{R}$ by $w(x,y) = v(x,-y)$. It follows from the chain rule that w is also harmonic. In the first case, we have that w equals v on ∂D, so the maximum principle implies that $w = v$ on $\overline{D}$. The same argument in the second case gives that $w = -v$ on $\overline{D}$. □

Let $D^+ = D_1 \cap \{y > 0\}$ denote the upper half-unit disk. We can now prove the reflection principle for harmonic functions.

Lemma 1.24. *If* $u : D^+ \to \mathbb{R}$ *is harmonic and* $u(x,0) = 0$, *then extending* u *to all of* D *by*

$$u(x,y) = -u(x,-y) \tag{1.121}$$

gives a harmonic function on all of D.

Proof. Define u on D by (1.121) and let $v : D \to \mathbb{R}$ be the harmonic function that is equal to u on the boundary.[3] We will show that $v = u$ on D.

Observe that the construction of u guarantees that $w(x,y) = -w(x,-y)$ on ∂D and, thus by (2) in Lemma 1.23, that $w(x,y) = -w(x,-y)$ on all of D. In particular, this gives that

$$w(x,0) = 0\,.$$

Therefore, w and u are harmonic functions on D^+ that agree on ∂D^+ (they obviously agree on $\partial D^+ \cap \partial D$ and both vanish on the remaining part of ∂D^+). By the maximum principle, $w = u$ on $\overline{D^+}$. The same argument applies in $D \setminus D^+$ and the lemma follows. □

A similar argument (see page 172 of [**A**]) shows that if f is holomorphic on D^+ and takes only real values on $\{y = 0\}$, it can be extended to a holomorphic function on all of D by setting

$$f(\bar{z}) = \overline{f(z)}\,.$$

Schwarz developed a version of the reflection principle for minimal surfaces and used it in the construction of triply periodic minimal surfaces. We will state a version from Lawson, [**La2**], and refer there for the proof:

Proposition 1.25 (Schwarz Reflection). *Suppose that* $u : D^+ \to \mathbb{R}^N$ *is a minimal surface and* u *maps* $\{y = 0\}$ *to a line* γ. *Then reflection across* γ *gives a minimal surface* $u : D \to \mathbb{R}^N$.

[3]The function u is given by solving the Dirichlet problem; see [**GiTr**], [**A**], or Chapter 4.

7. The Strong Maximum Principle

First note that the difference of two solutions of the minimal surface equation satisfies a uniformly elliptic divergence form equation (where the bound on the ellipticity depends on the bounds for the gradients of the minimal graphs):

Lemma 1.26. *If u_1 and u_2 are solutions of the minimal surface equation on a domain $\Omega \subset \mathbb{R}^n$, then $v = u_2 - u_1$ satisfies an equation of the form*

$$\operatorname{div}(a_{i,j}\, \nabla v) = 0\,, \tag{1.122}$$

where the eigenvalues of matrix $a_{i,j} = a_{i,j}(x)$ satisfy

$$0 < \mu \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1/\mu\,. \tag{1.123}$$

The constant μ depends only on the upper bounds for the gradients of $|\nabla u_i|$.

Proof. Define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(X) = \frac{X}{(1 + |X|^2)^{1/2}}\,. \tag{1.124}$$

By the fundamental theorem of calculus and the chain rule, we can write

$$\begin{aligned} F(\nabla u_2) - F(\nabla u_1) &= \int_0^1 \frac{d}{dt} \left(F(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\right) dt \\ &= \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\, \nabla(u_2 - u_1)\, dt \\ &= \left(\int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\, dt \right) \nabla(u_2 - u_1)\,. \end{aligned} \tag{1.125}$$

From this, we conclude that $v = u_2 - u_1$ satisfies an equation of the form

$$\operatorname{div}(a_{i,j}\, \nabla v) = 0\,, \tag{1.126}$$

where the matrix $a_{i,j}$ is given by (1.125).

Given a unit vector $V \in \mathbf{S}^{n-1}$ and $X \in \mathbb{R}^n$, we see that

$$dF(X)\, V = \frac{V}{(1 + |X|^2)^{1/2}} - \frac{\langle X, V \rangle}{(1 + |X|^2)^{3/2}}\, X\,. \tag{1.127}$$

In particular, taking the inner product with V gives

$$\begin{aligned} (1 + |X|^2)^{3/2} \langle V, dF(X)\, V \rangle &= (1 + |X|^2) - \langle X, V \rangle^2 \\ &\geq (1 + |X|^2) - |X|^2 = 1\,. \end{aligned} \tag{1.128}$$

It follows that $a_{i,j}$ is a weighted average of positive definite matrices and thus also positive definite, completing the proof. □

The following corollary is the local version of the strong maximum principle for minimal hypersurfaces:

Corollary 1.27. *Let $\Omega \subset \mathbb{R}^n$ be an open connected neighborhood of the origin. If u_1, $u_2 : \Omega \to \mathbb{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

Proof. Since the matrix $a_{i,j}$ given by Lemma 1.26 is positive definite, we can apply the maximum principle for linear equations to $v = u_2 - u_1$ (see, for instance, [**HaLi**] or Theorem 3.5 of [**GiTr**]). □

By writing a hypersurface locally as the graph of a function, we see that Corollary 1.27 has the following immediate consequence:

Corollary 1.28 (The Strong Maximum Principle). *If Σ_1, $\Sigma_2 \subset \mathbb{R}^n$ are complete connected minimal hypersurfaces (without boundaries), $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

As an application of the strong maximum principle, we next prove a generalization due to Schoen of a famous theorem of Rado (cf. Theorem 6.14).

Theorem 1.29 (Schoen, [**Sc2**]). *Let $\Omega \subset \mathbb{R}^2$ be strictly convex and $\sigma \subset \mathbb{R}^3$ a simple closed curve which is a graph over $\partial\Omega$ with bounded slope. Then any minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique.*

Proof. We show that Σ is graphical and leave the uniqueness to the reader. By the maximum principle, the interior of Σ is contained in the interior of the cylinder $\Omega \times \mathbb{R}$. Given a plane $\{x_3 = t\}$, we divide Σ into the portions Σ_t^+ above and Σ_t^- below the plane. Reflecting Σ_t^+ below the plane gives a new minimal surface $\tilde{\Sigma}_t^+$ below the plane. By the maximum principle, there cannot be a first t where Σ_t^- and $\tilde{\Sigma}_t^+$ have an interior point of contact. Since $\partial\Sigma$ is a graph, there cannot be a boundary point of contact. It follows then that the projection of Σ to the plane $\{x_3 = 0\}$ must be one to one, as desired. □

The combination of the maximum principle and reflection used above is known as "the method of moving planes". It was originally developed by Alexandrov to prove that a closed embedded constant mean curvature hypersurface in $\mathbb{R}^n$ must be a round sphere.

8. Second Variation Formula, Morse Index, and Stability

8.1. The second variation formula. Suppose now that $\Sigma^k \subset M^n$ is a minimal submanifold; we want to compute the second derivative of the area functional for a variation of Σ. Therefore, again let F be a variation of Σ

with compact support. In fact, we will assume that F is a normal variation, that is, on Σ we have

$$F_t^T \equiv 0\,. \tag{1.129}$$

As before, let x_i be local coordinates on Σ and set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j})\,, \tag{1.130}$$

$$\nu(t) = \sqrt{\det(g_{ij}(t))}\,\sqrt{\det(g^{ij}(0))}\,. \tag{1.131}$$

Differentiating the measure $\nu(t)$ gives

$$\frac{d^2}{dt^2_{t=0}} \mathrm{Vol}(F(\Sigma,t)) = \int \frac{d^2}{dt^2}_{t=0} \nu(t)\,\sqrt{\det(g_{ij}(0))}\,. \tag{1.132}$$

Recall that the first derivative of the measure $\nu(t)$ can be written as

$$2\,\frac{d}{dt}\nu(t) = \mathrm{Tr}(g'_{ij}(t) g^{\ell m}(t))\,\nu(t)\,, \tag{1.133}$$

where the trace here means $\sum_{i,j} g'_{ij}(t) g^{ij}(t)$. To see this, recall that

$$\frac{d}{dt_{t=0}} \det(\delta_{ij} + t\,a_{ij}) = \mathrm{Tr}(a_{ij})\,. \tag{1.134}$$

To evaluate $d^2/dt^2_{t=0}\nu(t)$ at some point $x \in \Sigma$, we may choose the coordinate system x_i to be orthonormal at x. Differentiating (1.133) then gives at x,

$$2\,\frac{d^2}{dt^2_{t=0}}\nu(t) = \mathrm{Tr}(g''_{ij}(0)) - \mathrm{Tr}(g'_{ij}(0)\,g'_{\ell m}(0)) + \frac{1}{2}\,[\mathrm{Tr}(g'_{ij}(0))]^2\,. \tag{1.135}$$

Since Σ is minimal, we have $\mathrm{Tr}(g'(0)) = 0$ and, therefore, we get

$$2\,\frac{d^2}{dt^2_{t=0}}\nu(t) = \mathrm{Tr}(g''(0)) - \mathrm{Tr}(g'(0)\,g'(0)) = \mathrm{Tr}(g''(0)) - |g'(0)|^2\,, \tag{1.136}$$

where the last equality used $\mathrm{Tr}(M^2) = |M|^2$ for a symmetric matrix M.

Lemma 1.30. *At the point x, we get*

$$|g'(0)|^2 = 4\,|\langle A(\cdot,\cdot), F_t\rangle|^2\,, \tag{1.137}$$

$$\begin{aligned}\mathrm{Tr}(g''(0)) = 2\,|\langle A(\cdot,\cdot), F_t\rangle|^2 + 2\,|\nabla_\Sigma^N F_t|^2 + 2\ \mathrm{Tr}\,\langle \mathrm{R}_M(\cdot\,, F_t)F_t, \cdot\rangle \\ + 2\ \mathrm{div}_\Sigma(F_{tt})\,.\end{aligned} \tag{1.138}$$

Proof. An easy calculation gives

$$g'_{ij}(0) = \langle F_{x_i t}, F_{x_j}\rangle + \langle F_{x_i}, F_{x_j t}\rangle = -2\,\langle A(F_{x_i}, F_{x_j}), F_t\rangle\,. \tag{1.139}$$

Since g_{ij} is the identity at x, the vectors F_{x_i} give an orthonormal basis for $T\Sigma$ at x and, thus, we get (1.137).

To get (1.138), we compute

$$\operatorname{Tr} g''(0) = 2\sum_{i=1}^{k} \langle F_{x_i tt}, F_{x_i}\rangle + 2\sum_{i=1}^{k}\langle F_{x_i t}, F_{x_i t}\rangle \,. \tag{1.140}$$

Next, use the definition[4] of the Riemann curvature tensor R_M of M to get

$$\begin{aligned}\sum_{i=1}^{k}\langle F_{x_i tt}, F_{x_i}\rangle &= \sum_{i=1}^{k}\langle \nabla_{F_t}\nabla_{F_t} F_{x_i}, F_{x_i}\rangle = \sum_{i=1}^{k}\langle \nabla_{F_t}\nabla_{F_{x_i}} F_t, F_{x_i}\rangle \\ &= \sum_{i=1}^{k}\langle \mathrm{R}_M(F_{x_i}, F_t)F_t, F_{x_i}\rangle + \sum_{i=1}^{k}\langle \nabla_{F_{x_i}}\nabla_{F_t}F_t, F_{x_i}\rangle \\ &= \sum_{i=1}^{k}\langle \mathrm{R}_M(F_{x_i}, F_t)F_t, F_{x_i}\rangle + \operatorname{div}_\Sigma(F_{tt})\,. \qquad (1.141)\end{aligned}$$

Combining these and using again that F_{x_i} is an orthonormal frame gives

$$\begin{aligned}\operatorname{Tr} g''(0) &= 2\sum_{i=1}^{k}\langle F^T_{x_i t}, F^T_{x_i t}\rangle + 2\sum_{i=1}^{k}\langle F^N_{x_i t}, F^N_{x_i t}\rangle \\ &\quad + 2\sum_{i=1}^{k}\langle \mathrm{R}_M(F_{x_i}, F_t)F_t, F_{x_i}\rangle + 2\operatorname{div}_\Sigma(F_{tt}) \\ &= 2|\langle A(\cdot,\cdot), F_t\rangle|^2 + 2|\nabla^N_\Sigma F_t|^2 + 2\operatorname{Tr}_\Sigma\langle \mathrm{R}_M(\cdot, F_t)F_t, \cdot\rangle + 2\operatorname{div}_\Sigma(F_{tt})\,.\end{aligned}$$

□

Therefore, we get at x,

$$\begin{aligned}\frac{d^2}{dt^2_{t=0}}\nu(t) &= -|\langle A(\cdot,\cdot), F_t\rangle|^2 \\ &\quad + |\nabla^N_\Sigma F_t|^2 - \operatorname{Tr}_\Sigma\langle \mathrm{R}_M(E_i, F_t)E_i, F_t\rangle + \operatorname{div}_\Sigma(F_{tt})\,.\end{aligned} \tag{1.142}$$

Note that we used the skew symmetry of R_M to reverse the sign. Observe that the right-hand side of (1.142) is independent of the choice of coordinates. Inserting (1.142) into (1.132), integrating and using the minimality of Σ and Stokes' theorem, we get

$$\begin{aligned}\frac{d^2}{dt^2_{t=0}}\operatorname{Vol}(F(\Sigma,t)) &= -\int_\Sigma |\langle A(\cdot,\cdot), F_t\rangle|^2 \\ &\quad + \int_\Sigma |\nabla^N_\Sigma F_t|^2 - \int_\Sigma \operatorname{Tr}_\Sigma\langle \mathrm{R}_M(\cdot, F_t)\cdot, F_t\rangle \\ &= -\int_\Sigma \langle F_t, L\,F_t\rangle\,. \qquad (1.143)\end{aligned}$$

[4] As on page 89 of [**dC2**], set $\mathrm{R}_M(U,V)W = \nabla_V\nabla_U W - \nabla_U\nabla_V W + \nabla_{[U,V]}W$.

The self-adjoint operator L is the so-called *stability operator* (or *Jacobi operator*) defined on a normal vector field X to Σ by

$$L\,X = \Delta_\Sigma^N\,X + \mathrm{Tr}\,[\mathrm{R}_M(\cdot,X)\,\cdot] + \tilde{A}(X)\,, \tag{1.144}$$

where $\tilde{A}$ is *Simons' operator* defined by

$$\tilde{A}(X) = \sum_{i,j=1}^{k} g(A(E_i,E_j),X)\,A(E_i,E_j) \tag{1.145}$$

and Δ_Σ^N is the *Laplacian on the normal bundle*, that is,

$$\Delta_\Sigma^N\,X = \sum_{i=1}^{k}(\nabla_{E_i}\nabla_{E_i}X)^N - \sum_{i=1}^{k}(\nabla_{(\nabla_{E_i}E_i)^T}X)^N\,. \tag{1.146}$$

A normal vector field X with $L\,X = 0$ is said to be a *Jacobi field.*

We will say that Σ has a *trivial normal bundle* if there is a global orthonormal basis for the normal bundle. When Σ is a hypersurface, this is equivalent to saying that Σ is two-sided.

For a hypersurface with a trivial normal bundle, the stability operator simplifies significantly since, in this case, it becomes an operator on functions. Namely, if we identify a normal vector field $X = \eta\,N$ with η, then

$$L\,\eta = \Delta_\Sigma\,\eta + |A|^2\,\eta + \mathrm{Ric}_M(N,N)\,\eta\,, \tag{1.147}$$

where Ric_M is the Ricci tensor of M (see, e.g., page 97 in [**dC2**]).

We will adopt the convention that λ is a (Dirichlet) eigenvalue of L on $\Omega \subset \Sigma$ if there exists a nontrivial normal vector field X which vanishes on $\partial\Omega$ so that

$$L\,X + \lambda X = 0\,. \tag{1.148}$$

Definition 1.31. The *Morse index* of a compact minimal surface Σ is the number of negative eigenvalues of the stability operator L (counted with multiplicity) acting on the space of smooth sections of the normal bundle which vanish on the boundary.

The second variation formula shows that if $\Sigma^k \subset M^n$ is a minimal submanifold, then the Hessian of the area functional at Σ is given by

$$-\int_\Sigma \langle\cdot, L\,\cdot\rangle\,. \tag{1.149}$$

It follows that we could have equivalently defined the Morse index of Σ to be the index of Σ as a critical point for the area functional.

8.2. Stability. We say that a minimal submanifold $\Sigma^k \subset M^n$ is *stable* if for all variations F with boundary fixed

$$\frac{d^2}{dt^2_{t=0}} \operatorname{Vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, L\, F_t\rangle \geq 0\,. \tag{1.150}$$

Observe that stability is the same as requiring the stability operator to be negative semidefinite (i.e., Morse index zero). Note also that if $\Sigma^{n-1} \subset \mathbb{R}^n$ is the graph of a function satisfying the minimal surface equation, then Σ is stable since Σ is, in fact, area-minimizing. A complete (possibly noncompact) minimal submanifold without boundary is said to be *stable* if all compact subdomains are stable.

For stable minimal hypersurfaces, we have the following useful inequality:

Lemma 1.32 (The Stability Inequality). *Suppose that $\Sigma^{n-1} \subset M^n$ is a stable minimal hypersurface with trivial normal bundle, then for all Lipschitz functions η with compact support*

$$\int_\Sigma (\inf_M \operatorname{Ric}_M + |A|^2)\, \eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2\,. \tag{1.151}$$

Proof. Since Σ is stable,

$$0 \leq -\int_\Sigma \eta\, L\eta = -\int_\Sigma (\eta\, \Delta_\Sigma\, \eta + |A|^2\, \eta^2 + \operatorname{Ric}_M(N,N)\, \eta^2)\,. \tag{1.152}$$

Integrating by parts gives

$$\int_\Sigma (\operatorname{Ric}_M(N,N) + |A|^2)\, \eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2\,. \tag{1.153}$$

This proves the lemma. □

The stability inequality gives restrictions on a stable minimal hypersurface when we have some positivity of the curvature of M. The next corollary records two versions of this (the first due to J. Simons and the second to Schoen and Yau):

Corollary 1.33. *Suppose that $\Sigma^{n-1} \subset M^n$ is a closed stable minimal hypersurface with trivial normal bundle.*

- *If* $\operatorname{Ric}_M \geq 0$*, then Σ is totally geodesic and* $\operatorname{Ric}_M(N,N) = 0$ *on Σ.*
- *If* $\operatorname{Scal}_M > 0$ *and $n = 3$, then Σ is an $\mathbf{S}^2$ or an $\mathbb{RP}^2$ and*

$$\int_\Sigma (\operatorname{Scal}_M + |A|^2) \leq 8\,\pi\,. \tag{1.154}$$

Proof. Since Σ is compact and has no boundary, we can use the constant function $\eta = 1$ in the stability inequality to get

$$\int_\Sigma (\mathrm{Ric}_M(N,N) + |A|^2) \leq 0 \,. \tag{1.155}$$

The first conclusion follows immediately. For the second, let

$$E_1,\ E_2,\ \text{and } E_3 = N$$

be an orthonormal basis along Σ. Since $R_{3333} = 0$, the normal Ricci curvature is given by

$$\mathrm{Ric}_M(N,N) = R_{1313} + R_{2323} \,. \tag{1.156}$$

Similarly, we have $\mathrm{Ric}_M(E_1,E_1) = R_{1212} + R_{1313}$ and $\mathrm{Ric}_M(E_2,E_2) = R_{1212} + R_{2323}$. Thus, the scalar curvature is given by

$$\mathrm{Scal}_M = \mathrm{Ric}_M(E_1,E_1) + \mathrm{Ric}_M(E_2,E_2) + \mathrm{Ric}_M(N,N) \tag{1.157}$$

$$= 2\,(R_{1212} + R_{1313} + R_{2323}) = 2\ \mathrm{Ric}_M(N,N) + 2\,R_{1212} \,. \tag{1.158}$$

The Gauss equation and minimality give that $K_\Sigma = R_{1212} + \det(A) = R_{1212} - |A|^2/2$, so we get that

$$\mathrm{Ric}_M(N,N) = 1/2\ \mathrm{Scal}_M - \mathrm{K}_\Sigma - 1/2\,|A|^2 \,. \tag{1.159}$$

Substituting this into (1.155), we get that

$$1/2 \int_\Sigma (\mathrm{Scal}_M + |A|^2) \leq \int_\Sigma \mathrm{K}_\Sigma = 2\,\pi\,\chi(\Sigma) \,, \tag{1.160}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and the equality used the Gauss-Bonnet formula (see page 274 of [**dC1**]). It follows that $\chi(\Sigma) > 0$, giving the claim. □

By using Lemma 1.32 and a logarithmic cutoff argument (as in the proof of Theorem 1.21), it is easy to give a second proof of Theorem 1.21. We will return to this point of view in the next chapter.

8.3. A characterization of stability. We will close this section with some useful characterizations of stability for minimal hypersurfaces with trivial normal bundle and we will derive some consequences. This will require more background in PDE than in the rest of these notes; when this occurs, precise references will be given.

For minimal hypersurfaces with trivial normal bundle, we saw that stability was equivalent to $\lambda_1(\Omega, L) \geq 0$ for every $\Omega \subset \Sigma$ where

$$\lambda_1(\Omega, L) = \inf \left\{ - \int \eta\, L\eta \mid \eta \in C_0^\infty(\Omega) \text{ and } \int_\Omega \eta^2 = 1 \right\} . \tag{1.161}$$

For smooth functions u, we define the $W^{1,2}$-norm by

$$\|u\|_{W^{1,2}}^2 = \int u^2 + \int |\nabla u|^2 . \tag{1.162}$$

The Sobolev space $W_0^{1,2}(\Omega)$ is the closure of the space of compactly supported smooth functions on Ω with respect to the $W^{1,2}$-norm. Similarly, $W^{1,2}(\Omega)$ is the closure of the space of smooth functions on Ω with respect to the $W^{1,2}$-norm.

By standard elliptic theory, see, for instance, [**HaLi**] or [**GiTr**], we get the following:

Lemma 1.34. *If L and $\Omega \subset \Sigma$ are as above and $\lambda_1 = \lambda_1(\Omega, L)$, then*

$$\lambda_1 = \inf \left\{ \frac{\int |\nabla_\Sigma \eta|^2 - |A|^2 \eta^2 - \mathrm{Ric}_M(N,N)\eta^2}{\int \eta^2} \mid \eta \in W_0^{1,2}(\Omega) \right\} . \tag{1.163}$$

Moreover, if $u \in W_0^{1,2}(\Omega)$ satisfies

$$\frac{\int |\nabla_\Sigma u|^2 - |A|^2 u^2 - \mathrm{Ric}_M(N,N)u^2}{\int u^2} = \lambda_1 , \tag{1.164}$$

then u is smooth and $Lu = -\lambda_1 u$.

Proof. Set I equal to the right-hand side of (1.163). To keep the notation short, define the function V on Σ by

$$V(x) = |A|^2(x) + \mathrm{Ric}_M(N,N)(x) . \tag{1.165}$$

Since the divergence theorem gives that $\int |\nabla_\Sigma \eta|^2 = - \int \eta \, \Delta_\Sigma \eta$ when η is C^2 with compact support, we always have that

$$I \leq \lambda_1 . \tag{1.166}$$

For the other direction, choose a minimizing sequence $\eta_j \in W_0^{1,2}(\Omega)$ with

$$\frac{\int |\nabla_\Sigma \eta_j|^2 - V \eta_j^2}{\int \eta_j^2} \leq I + \frac{1}{j} . \tag{1.167}$$

After multiplying η_j by a constant, we can assume that $\int \eta_j^2 = 1$. By Rellich's compactness theorem (theorem 7.22 in [**GiTr**]), we can pass to a subsequence (still denoted by η_j) so that:

- η_j converges weakly in $W^{1,2}$ to a function $\eta \in W_0^{1,2}$.
- η_j converges strongly in L^2 to η and, in particular, $\int \eta^2 = 1$.

Moreover, energy is lower semi-continuous under weak convergence, so that

$$\int |\nabla_\Sigma \eta|^2 \leq \liminf \int |\nabla_\Sigma \eta_j|^2 .$$

Using that V is a continuous function, it follows that η satisfies

$$\frac{\int |\nabla_\Sigma \eta|^2 - V\,\eta^2}{\int \eta^2} \leq I\,. \tag{1.168}$$

By the definition of I, we must have equality in (1.168) and, in particular, η realizes the infimum I.

Both claims will follow from showing that any $\eta \in W_0^{1,2}$ that satisfies equality in (1.168) is smooth and satisfies $L\,\eta = -I\,\eta$. We prove this next. Suppose therefore that $\eta \in W_0^{1,2}$ satisfies equality in (1.168) and is normalized so that

$$\int \eta^2 = 1\,.$$

If ψ is a smooth function with compact support in Ω and

$$\int \psi\eta = 0\,, \tag{1.169}$$

then, obviously,

$$\frac{d}{dt}_{t=0} \int (\eta + t\psi)^2 = 0\,. \tag{1.170}$$

By (1.170) and the definition of I, we must have

$$\begin{aligned} 0 &= \frac{d}{dt}_{t=0} \int |\nabla_\Sigma(\eta + t\psi)|^2 - V\,(\eta + t\,\psi)^2 \\ &= 2 \int \langle \nabla_\Sigma \eta, \nabla_\Sigma \psi\rangle - V\,\eta\,\psi\,. \end{aligned} \tag{1.171}$$

By approximation, equation (1.171) holds for any $\psi \in W_0^{1,2}(\Omega)$ satisfying (1.169). In particular, given any $\phi \in W_0^{1,2}(\Omega)$, then (1.171) holds for

$$\psi = \phi - \eta \int (\phi\eta)\,, \tag{1.172}$$

and thus

$$\int \langle \nabla_\Sigma \eta, \nabla_\Sigma \phi\rangle - V\,\eta\,\phi = \left(\int (\phi\eta)\right) \int |\nabla_\Sigma \eta|^2 - V\,\eta^2 = I \int (\phi\eta)\,, \tag{1.173}$$

where the last equality used that η satisfies equality in (1.168) and $\int \eta^2 = 1$. Since (1.173) holds for all $\phi \in W_0^{1,2}(\Omega)$, η is a weak solution to

$$L\eta = (\Delta_\Sigma + V)\,\eta = -I\,\eta\,.$$

Thus, by elliptic regularity (theorem 8.8 of [**GiTr**]), η must be smooth and must satisfy $L\,\eta = -I\,\eta$. Since η is smooth, it is a valid test function in the definition of λ_1, so we see that

$$\lambda_1 \leq I\,. \tag{1.174}$$

Combining this with the (easy) opposite inequality shows that $\lambda_1 = I$, giving the first claim of the lemma. Since we have already shown that η is smooth and $L\eta = -I\,\eta$, the second claim now also follows. $\square$

Combining Lemma 1.34 and the Harnack inequality, we see in the next lemma that any eigenfunction for the first eigenvalue cannot change sign.

Lemma 1.35. *If u is a smooth function on Ω that vanishes on $\partial\Omega$ and $Lu = -\lambda_1 u$ where $\lambda_1 = \lambda_1(\Omega, L)$, then u cannot change sign in Ω.*

Proof. We may assume that u is not identically zero. Since u vanishes on $\partial\Omega$, so does $|u|$. In fact, it is easy to see that $|u|$ also achieves the minimum in (1.164) and hence, by Lemma 1.34, we have that $|u|$ is smooth and $L\,|u| = -\lambda_1\,|u|$. Since $|u| \geq 0$ and $|u|$ is not identically zero, the Harnack inequality, Lemma 1.46, implies that $|u| > 0$ in Ω and the lemma follows. $\square$

Since the eigenfunctions with different eigenvalues are all orthogonal to each other, Lemma 1.35 implies that only the lowest eigenfunction does not change sign and, in fact, the first eigenvalue has multiplicity one. As a consequence, we see that if $\Sigma \subset M$ is a stable minimal hypersurface with trivial normal bundle and without boundary, then $\tilde{\Sigma} \subset \tilde{M}$ is also stable where

$$G : \tilde{M} \to M \tag{1.175}$$

is a covering map, $\tilde{\Sigma} = G^{-1}(\Sigma)$, and $\tilde{M}$ is given the pullback metric. On the other hand, easy examples show that a cover of a stable minimal submanifold is not in general stable (consider, for instance, $\mathbb{RP}^2 \subset \mathbb{RP}^3$).

More generally, we have the following version of Barta's theorem, [**Ba**]:

Lemma 1.36. *Let Σ be a minimal hypersurface with trivial normal bundle, L its stability operator, and $\Omega \subset \Sigma$ a bounded domain. If there exists a positive function u on Ω with $Lu = 0$, then Ω is stable.*

Proof. Set $q = |A|^2 + \mathrm{Ric}_M(N, N)$ so that $L = \Delta_\Sigma + q$. Since $u > 0$, $w = \log u$ is well defined and satisfies

$$\Delta_\Sigma w = -q - |\nabla_\Sigma w|^2 . \tag{1.176}$$

Let f be a compactly supported smooth function on Ω. Multiplying both sides of (1.176) by f^2 and integrating by parts gives

$$\begin{aligned} \int f^2\, q + \int f^2\, |\nabla_\Sigma w|^2 &= -\int f^2\, \Delta_\Sigma w \leq 2 \int |f|\, |\nabla_\Sigma f|\, |\nabla_\Sigma w| \\ &\leq \int f^2\, |\nabla_\Sigma w|^2 + \int |\nabla_\Sigma f|^2 , \end{aligned} \tag{1.177}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Cancelling the $\int f^2 |\nabla_\Sigma w|^2$ term from both sides of (1.177), we see that

$$-\int f\,Lf \geq 0\,. \tag{1.178}$$

Since this is true for any such f, the lemma follows. □

Note that if Σ is closed or, more generally, if u vanishes on $\partial\Sigma$, then Lemma 1.36 follows immediately from Lemma 1.35.

The variation $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ given by

$$F(\cdot, t) : (x_1, x_2, x_3) \to (x_1, x_2, x_3 + t) \tag{1.179}$$

is a one-parameter group of isometries of $\mathbb{R}^3$, and hence for any surface $\Sigma \subset \mathbb{R}^3$ we have that Area $F(\Sigma, t)$ is constant. The variation vector field is $F_t = (0, 0, 1)$. If Σ is minimal, then the second variation formula implies that $\langle N, (0, 0, 1)\rangle$ (the normal component of the variation vector field) is a Jacobi field. Furthermore, when $\Sigma = \text{Graph}_u$

$$\langle N, (0, 0, 1)\rangle = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{R}^2} u|^2}}\,, \tag{1.180}$$

is therefore a positive Jacobi field. Consequently, Lemma 1.36 gives another way to see that minimal graphs are stable.

A manifold Σ is said to be *parabolic* if any positive superharmonic function u (i.e., $\Delta_\Sigma u \leq 0$) is constant. The next proposition uses the logarithmic cutoff trick to see that quadratic volume growth implies parabolicity. In this proposition, we will let $B_s^\Sigma = B_s^\Sigma(p)$ denote an intrinsic (geodesic) ball in Σ.

Proposition 1.37. *If Σ is a complete surface so that for all $s > 0$ we have*

$$\text{Area}\,(B_s^\Sigma) \leq C\, s^2\,, \tag{1.181}$$

then Σ is parabolic.

Proof. Suppose that $u > 0$ and $\Delta_\Sigma u \leq 0$ and set

$$w = \log u\,, \tag{1.182}$$

so that $|\nabla_\Sigma w|^2 \leq -\Delta_\Sigma w$. Let r denote the distance to p and define the cutoff function η by

$$\eta = \begin{cases} 1 & \text{if } r^2 \leq R\,, \\ 2 - \frac{\log r^2}{\log R} & \text{if } R < r^2 \leq R^2\,, \\ 0 & \text{if } r^2 > R^2\,. \end{cases} \tag{1.183}$$

By Stokes' theorem and the absorbing inequality ($2ab \leq \frac{1}{2}a^2 + 2b^2$), we get

$$\int \eta^2 \, |\nabla_\Sigma w|^2 \leq - \int \eta^2 \, \Delta_\Sigma w \leq 2 \int \eta \, |\nabla_\Sigma \eta| \, |\nabla_\Sigma w| \tag{1.184}$$
$$\leq \frac{1}{2} \int \eta^2 \, |\nabla_\Sigma w|^2 + 2 \int |\nabla_\Sigma \eta|^2 .$$

Substituting the definition of η and the area bound gives

$$\int_{B^\Sigma_{\sqrt{R}}} |\nabla_\Sigma w|^2 \leq \int \eta^2 \, |\nabla_\Sigma w|^2 \leq 4 \int |\nabla_\Sigma \eta|^2$$
$$\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} \int_{(B^\Sigma_{\mathrm{e}^\ell} \setminus B^\Sigma_{\mathrm{e}^{\ell-1}})} r^{-2} \tag{1.185}$$
$$\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} C \, \mathrm{e}^2 \leq \frac{8 \, C \, \mathrm{e}^2}{\log R} .$$

Letting $R \to \infty$, we get that w is constant. □

Applying Proposition 1.37, we see that an entire two-dimensional minimal graph Σ is parabolic. This follows since the intrinsic distance is bounded from below by the Euclidean distance and therefore the area bound (1.20) implies that minimal graphs have quadratic area growth. Setting

$$u = \langle N, (0,0,1) \rangle \tag{1.186}$$

as in (1.180) gives a positive Jacobi field. In particular,

$$\Delta_\Sigma u = -(|A|^2 + \mathrm{Ric}_{\mathbb{R}^3}(N,N)) \, u = - |A|^2 \, u \leq 0 , \tag{1.187}$$

so that u is a positive superharmonic function. By Proposition 1.37, u must be constant so that $\Delta_\Sigma u = 0$ and hence $|A|^2 = 0$. In other words, any complete minimal graph defined on $\mathbb{R}^2$ must be flat. This yields another proof of the Bernstein theorem, Theorem 1.21.

Remark 1.38. *It follows from Osserman's theorem that all minimal surfaces of finite total curvature are parabolic. The helicoid and the classical examples of Riemann have infinite total curvature, but are also parabolic.*

We will next give a characterization of stability for complete noncompact minimal hypersurfaces with trivial normal bundle due to Fischer-Colbrie and Schoen (cf. [**Ba**] and [**Du**]). We will assume that the boundary is smooth if it is nonempty.

Proposition 1.39 (Fischer-Colbrie and Schoen, [**FiSc**]). *If Σ is a complete noncompact minimal hypersurface with trivial normal bundle, then the*

following are equivalent:

(1.188) $\lambda_1(\Omega, L) \geq 0$ *for every bounded domain* $\Omega \subset \Sigma$.

(1.189) $\lambda_1(\Omega, L) > 0$ *for every bounded domain* $\Omega \subset \Sigma$.

(1.190) *There exists a positive function* u *with* $L\,u = 0$.

Proof. By Lemma 1.36, (1.190) implies (1.188).

Clearly (1.189) implies (1.188). To see the equivalence of (1.188) and (1.189), given any bounded domain $\Omega_0 \subset \Sigma$ choose a strictly larger bounded domain Ω_1. The variational characterization of eigenvalues, (1.161), implies that

$$\lambda_1(\Omega_0, L) \geq \lambda_1(\Omega_1, L) \geq 0\,, \tag{1.191}$$

where the second inequality follows from (1.188). Let u_0 denote the first eigenfunction for L on Ω_0, and define u_1 on Ω_1 by

$$u_1(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega_0\,, \\ 0 & \text{otherwise}. \end{cases} \tag{1.192}$$

If we had equality in (1.191), then, by Lemma 1.34, $L\,u_1 = -\lambda_1 u_1$ on Ω_1 and, by Lemma 1.35, $u_1 > 0$ on Ω_1. This is not possible since u_1 vanishes on $\Omega_1 \setminus \Omega_0$, and thus the equivalence of (1.188) and (1.189) follows.

It remains to show that (1.189) implies (1.190). To do this, fix $p \in \Sigma$ and for each $r > 0$ let

$$B_r^\Sigma = B_r^\Sigma(p) = \{q \in \Sigma \mid \operatorname{dist}_\Sigma(p, q) < r\}\,. \tag{1.193}$$

Since $\lambda_1(B_r^\Sigma, L) > 0$, by the Fredholm alternative (see theorem 6.15 of [**GiTr**]), there exists a unique function v_r with

$$L\,v_r = -|A|^2 - \operatorname{Ric}_M(N, N) \text{ on } B_r^\Sigma \quad \text{and} \quad v_r = 0 \text{ on } \partial B_r^\Sigma\,. \tag{1.194}$$

Setting $u_r = v_r + 1$, (1.194) gives

$$L\,u_r = 0 \text{ on } B_r^\Sigma \quad \text{and} \quad u_r = 1 \text{ on } \partial B_r^\Sigma\,. \tag{1.195}$$

We claim that

$$u_r > 0 \text{ on } B_r^\Sigma\,. \tag{1.196}$$

By the Harnack inequality, Lemma 1.46, it suffices to show that $u_r \geq 0$ in B_r^Σ. If this fails, then we can choose a nonempty connected component Ω of the open set

$$\{x \in B_r^\Sigma \mid u_r(x) < 0\}\,. \tag{1.197}$$

By construction, we have $u_r < 0$ in Ω and $u_r = 0$ on $\partial\Omega$, and Lemma 1.35 implies that $\lambda_1(\Omega, L) = 0$. This contradiction implies that (1.196) holds.

For each r, we define a positive function w_r by

$$w_r = (u_r(p))^{-1}\, u_r \tag{1.198}$$

and observe that $Lw_r = 0$ and $w_r(p) = 1$.

Now, let K be any compact set with $K \subset B^{\Sigma}_{R_0}$. Applying the Harnack inequality (see theorem 8.27 of [**GiTr**] for the estimates up to $\partial\Sigma$), we get for any $r \geq 2R_0$ that

$$\sup_K w_r \leq C_K\,. \tag{1.199}$$

The interior and boundary Schauder estimates (theorems 6.2 and 6.6 of [**GiTr**]) imply that

$$|w_r|_{C^{2,\alpha}_K} \leq C'_K\,. \tag{1.200}$$

In other words, if $K \subset B_{R_0^{\Sigma}}$, we have uniform $C^{2,\alpha}_K$ estimates for every w_r for $r \geq 2R_0$. By the Arzela-Ascoli theorem, we can choose a subsequence of the w_r that converges uniformly in $C^{2,\frac{\alpha}{2}}$ on compact sets to a function w. This convergence guarantees that w satisfies $Lw = 0$. Since each w_r was positive and $w_r(p) = 1$, w is nonnegative and has $w(p) = 1$. Finally, the Harnack inequality implies that w is also positive, which completes the proof. $\square$

We can use Proposition 1.39 to give a slight generalization of the Bernstein theorem.

Corollary 1.40. *If $\Sigma \subset \mathbb{R}^3$ is a complete, connected, stable, parabolic, 2-sided minimal surface without boundary, then it must be a plane.*

Proof. Since Σ is orientable and stable, Proposition 1.39 implies that there exists a function $u > 0$ with

$$\Delta_\Sigma u = -(|A|^2 + \mathrm{Ric}_{\mathbb{R}^3}(N,N))\, u = -\,|A|^2\, u \leq 0\,. \tag{1.201}$$

Since Σ is parabolic, u must be constant. Hence (1.201) implies that $|A| \equiv 0$ and the corollary follows. $\square$

9. Multi-valued Graphs

The simplest minimal surfaces are minimal graphs, where the entire surface can be written as the graph of a function over a domain in a plane. More generally, each point in a surface could have a neighborhood where it is a graph over a fixed plane, but the projection from the surface to the plane may not be one-to-one. In this case, the surface fails to be a graph, but is instead a "multi-valued graph". The analysis of embedded multi-valued graphs played an important role in [**CM2**]–[**CM7**].

Intuitively, an (embedded) multi-valued graph is a surface such that over each point of an annulus in the plane, the surface consists of N graphs. To

make this notion precise, let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. Following [**CM2**], an *N-valued graph* on the annulus $D_s \setminus D_r$ is a single valued graph of a function u over

$$\{(\rho,\theta) \mid r < \rho \leq s\,,\, |\theta| \leq N\,\pi\}\,. \tag{1.202}$$

For working purposes, we generally think of the intuitive picture of a multi-sheeted surface in $\mathbb{R}^3$, and we identify the single-valued graph over the universal cover with its multi-valued image in $\mathbb{R}^3$.

The multi-valued graphs that we will consider will all be embedded, which corresponds to a nonvanishing separation between the sheets (or the floors). Following [**CM2**], the *separation* is the function (see Figure 1.25)

$$w(\rho,\theta) = u(\rho,\theta + 2\pi) - u(\rho,\theta)\,. \tag{1.203}$$

If Σ is the helicoid, then $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞-valued graphs on $\mathbf{C} \setminus \{0\}$. Σ_1 is the graph of the function $u_1(\rho,\theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho,\theta) = \theta + \pi$. ($\Sigma_1$ is the subset where $s > 0$ in (1.46) and Σ_2 the subset where $s < 0$.) In either case the separation $w = 2\,\pi$. A *multi-valued minimal graph* is a multi-valued graph of a function u satisfying the minimal surface equation.

Note that for an embedded multi-valued graph, the sign of w determines whether the multi-valued graph spirals in a left-handed or right-handed manner, in other words, whether upwards motion corresponds to turning in a clockwise direction or in a counterclockwise direction.

It is easy to see that a multi-valued graph must be stable. Namely, the function $\langle N, (0,0,1)\rangle$ is a nonvanishing Jacobi field so stability follows immediately from Lemma 1.36.

Finally, we note that the separation w satisfies a reasonably nice divergence form equation. Namely, Lemma 1.26 implies that

$$\operatorname{div}(a_{i,j}\,\nabla w) = 0\,, \tag{1.204}$$

where the matrix $a_{i,j}$ is bounded and positive definite. In fact, the eigenvalues of $a_{i,j}$ are uniformly bounded in terms of the maximum of the gradient of the multi-valued function u. Note that this is enough to get a Harnack inequality for w so long as the gradient of u is uniformly bounded.

10. Local Examples of Multi-valued Graphs

We will use the Weierstrass representation to construct a sequence of disks

$$\Sigma_i \subset B_1 = B_1(0) \subset \mathbb{R}^3 \tag{1.205}$$

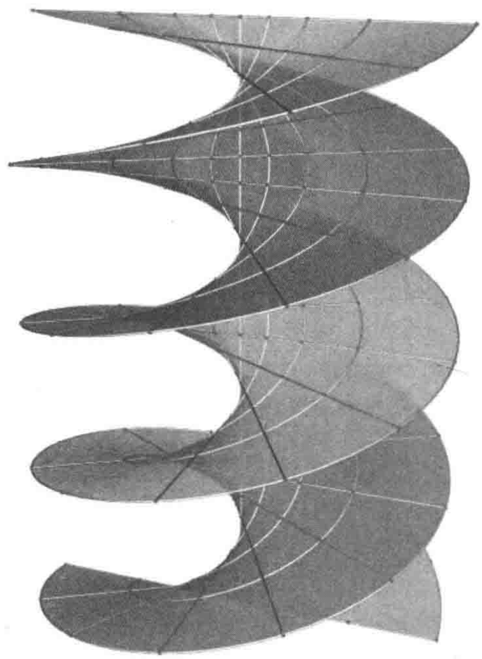

Figure 1.24. Multi-valued graphs. The helicoid is obtained by gluing together two ∞-valued graphs along a line. Credit: Matthias Weber, www.indiana.edu/~minimal.

where the curvatures blow up only at 0 (see (1) and (2) in Theorem 1.41 below) and $\Sigma_i \setminus \{x_3\text{-axis}\}$ consists of two multi-valued graphs for each i; see (3). Furthermore (see (4)), $\Sigma_i \setminus \{x_3 = 0\}$ converges to two embedded minimal disks $\Sigma^- \subset \{x_3 < 0\}$ and $\Sigma^+ \subset \{x_3 > 0\}$ each of which spirals into $\{x_3 = 0\}$ and thus is not proper; see Figure 1.26.

Theorem 1.41 (Colding-Minicozzi, [**CM18**]). *There is a sequence of compact embedded minimal disks $0 \in \Sigma_i \subset B_1 \subset \mathbb{R}^3$ with $\partial \Sigma_i \subset \partial B_1$ and containing the vertical segment $\{(0,0,t) \mid |t| < 1\} \subset \Sigma_i$ so:*

(1) $\lim_{i \to \infty} |A_{\Sigma_i}|^2(0) = \infty$.

(2) $\sup_i \sup_{\Sigma_i \setminus B_\delta} |A_{\Sigma_i}|^2 < \infty$ *for all* $\delta > 0$.

(3) $\Sigma_i \setminus \{x_3\text{-}axis\} = \Sigma_{1,i} \cup \Sigma_{2,i}$ *for multi-valued graphs* $\Sigma_{1,i}$ *and* $\Sigma_{2,i}$.

(4) $\Sigma_i \setminus \{x_3 = 0\}$ *converges to two embedded minimal disks* $\Sigma^\pm \subset \{\pm x_3 > 0\}$ *with* $\overline{\Sigma^\pm} \setminus \Sigma^\pm = B_1 \cap \{x_3 = 0\}$. *Moreover,* $\Sigma^\pm \setminus \{x_3\text{-}axis\} = \Sigma_1^\pm \cup \Sigma_2^\pm$ *for multi-valued graphs* $\Sigma_1^\pm$ *and* $\Sigma_2^\pm$ *each of which spirals into* $\{x_3 = 0\}$*; see Figure* 1.26.

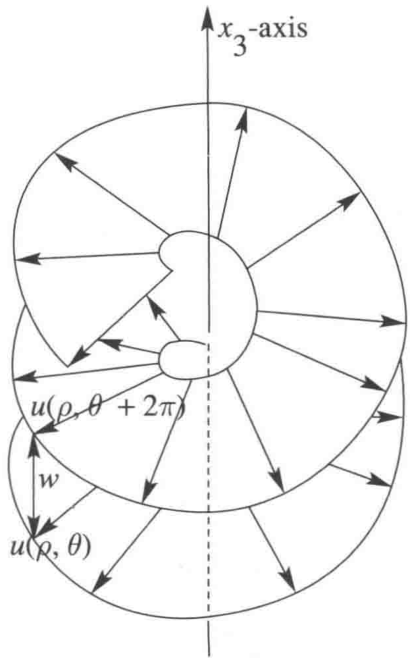

Figure 1.25. The separation w for a multi-valued minimal graph; see (1.203).

It follows from (4) that $\Sigma_i \setminus \{0\}$ converges to a lamination of $B_1 \setminus \{0\}$ (with leaves Σ^-, Σ^+, and $B_1 \cap \{x_3 = 0\} \setminus \{0\}$) which does not extend to a lamination of B_1. Namely, 0 is not a removable singularity.

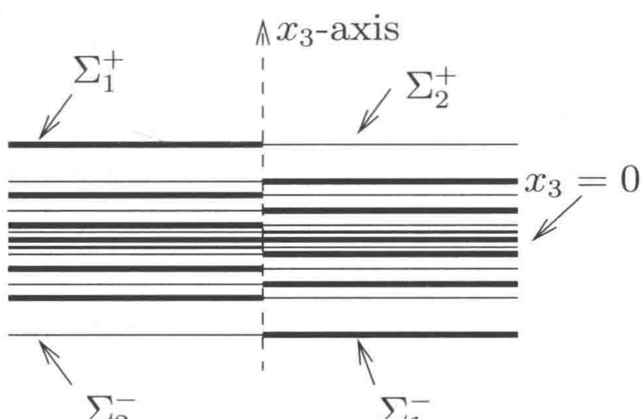

Figure 1.26. A schematic picture of the examples in Theorem 1.41. Removing the x_3-axis disconnects the surface into two multi-valued graphs; one of these is in bold.

To show Theorem 1.41, we first construct a one-parameter family (with parameter $a \in (0, 1/2)$) of minimal immersions F_a by making a specific

choice of Weierstrass data $g = e^{ih_a}$ (where $h_a = u_a + i\,v_a$), $\phi = dz$, and domain Ω_a to use in (1.109). We show in Lemma 1.43 that this one-parameter family of immersions is compact. Lemma 1.44 shows that the immersions $F_a : \Omega_a \to \mathbb{R}^3$ are embeddings.

The next lemma records the differential of F.

Lemma 1.42. *If F is given by* (1.109) *with $g(z) = e^{i\,(u(z)+iv(z))}$ and $\phi = dz$, then*

$$\partial_x F = (\sinh v\, \cos u, \sinh v\, \sin u, 1)\,, \tag{1.206}$$

$$\partial_y F = (\cosh v\, \sin u, -\cosh v\, \cos u, 0)\,. \tag{1.207}$$

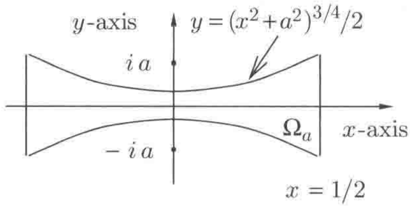

Figure 1.27. The domain Ω_a.

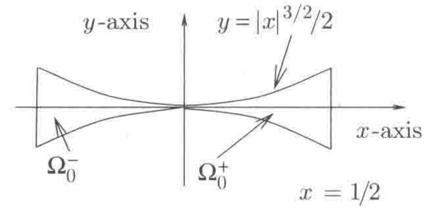

Figure 1.28. $\Omega_0 = \bigcap_{a>0} \Omega_a \setminus \{0\}$ and its two components Ω_0^+ and Ω_0^-.

For each $0 < a < 1/2$, set (see Figure 1.27)

$$h_a(z) = \frac{1}{a}\ \arctan\left(\frac{z}{a}\right) \tag{1.208}$$

on the domain

$$\Omega_a = \{(x,y)\,|\,|x| \le 1/2,\ |y| \le (x^2+a^2)^{3/4}/2\}\,. \tag{1.209}$$

Note that h_a is well defined since Ω_a is simply connected and $\pm i\,a \notin \Omega_a$. For future reference

$$\partial_z h_a(z) = \frac{1}{z^2+a^2} = \frac{x^2+a^2-y^2-2i\,xy}{(x^2+a^2-y^2)^2+4x^2y^2}\,, \tag{1.210}$$

$$\mathrm{K}_a(z) = \frac{-|\partial_z h_a|^2}{\cosh^4 v_a} = \frac{-|z^2+a^2|^{-2}}{\cosh^4(\mathrm{Im}\ \arctan(z/a)/a)}\,. \tag{1.211}$$

Here (1.211) used (1.117). Note that, by the Cauchy-Riemann equations,

$$\begin{aligned} \partial_z h_a &= (\partial_x - i\,\partial_y)(u_a + iv_a)/2 = \partial_x u_a - i\,\partial_y u_a \\ &= \partial_y v_a + i\,\partial_x v_a\,. \end{aligned} \tag{1.212}$$

In the rest of this section, we let $F_a : \Omega_a \to \mathbb{R}^3$ be from (1.109) with $g = e^{i\,h_a}$, $\phi = dz$, and $z_0 = 0$. Set $\Omega_0 = \cap_a \Omega_a \setminus \{0\}$ so that

$$\Omega_0 = \{(x,y)\,|\,0 < |x| \le 1/2,\ |y| \le |x|^{3/2}/2\}\,. \tag{1.213}$$

(See Figure 1.28.) The family of functions h_a is not compact since

$$\lim_{a\to 0} |h_a|(z) = \infty$$

for $z \in \Omega_0$. However, the next lemma shows that the family of immersions F_a is compact.

Lemma 1.43. *If $a_j \to 0$, then there is a subsequence a_i so F_{a_i} converges uniformly in C^2 on compact subsets of Ω_0.*

Proof. Since h_a and $-1/z$ are holomorphic and

$$|\partial_z h_a(z) - \partial_z(-1/z)| = \frac{a^2}{|z|^2\,|z^2+a^2|}, \tag{1.214}$$

we get easily that

$$\lim_{a\to 0} \nabla h_a = \nabla(-1/z), \tag{1.215}$$

where the convergence is uniform on compact subsets of Ω_0. Since each $v_a(x,0) = 0$, the fundamental theorem of calculus gives that the v_a's converge uniformly in C^1 on compact subsets of Ω_0. (Unfortunately, the u_a's do not converge.)

Let $\Omega_0^{\pm} = \{\pm x > 0\} \cap \Omega_0$ be the two components of Ω_0; see Figure 1.28. Set $b_j^+ = u_{a_j}(1/2)$ and $b_j^- = u_{a_j}(-1/2)$ and choose a subsequence a_i so both b_i^- and b_i^+ converge modulo 2π (this is possible since $T^2 = \mathbb{R}^2/(2\pi\mathbf{Z}^2)$ is compact). Arguing as above, $h_{a_i} - b_i^{\pm}$ converges uniformly in C^1 on compact subsets of $\Omega_0^{\pm}$. Therefore, by Lemma 1.42, the minimal immersions corresponding to Weierstrass data $g = e^{i(h_{a_i} - b_i^{\pm})}$, $\phi = dz$ converge uniformly in C^2 on compact subsets of $\Omega_0^{\pm}$ as $i \to \infty$. □

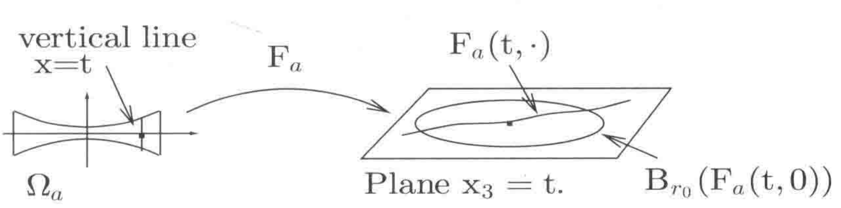

Figure 1.29. A horizontal slice in Lemma 1.44.

The main difficulty in proving Theorem 1.41 is showing that the immersions $F_a : \Omega_a \to \mathbb{R}^3$ are embeddings. This will follow easily from (A) and (B) below. Namely, we show in Lemma 1.44 (see Figures 1.29 and 1.30) that for $|t| \le 1/2$:

(A) The horizontal slice $\{x_3 = t\} \cap F_a(\Omega_a)$ is the image of the vertical segment $\{x = t\}$ in the plane, i.e., $x_3(F_a(x,y)) = x$; see (P1).

(B) The image $F_a(\{x=t\}\cap\Omega_a)$ is a graph over a line segment in the plane $\{x_3=t\}$ (the line segment will depend on t); see (P2).

(C) The boundary of the graph in (B) is outside the ball $B_{r_0}(F_a(t,0))$ for some $r_0>0$ and all a; see (P3).

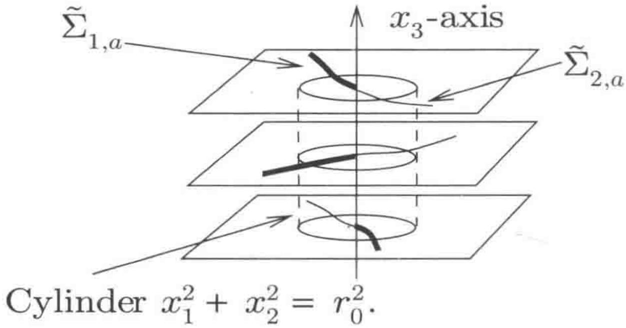

Figure 1.30. Horizontal slices of $F_a(\Omega_a)$ in Lemma 1.44.

Lemma 1.44. *The immersions F_a have the following properties:*

(P1) *For each x, y, we have $x_3(F_a(x,y))=x$.*

(P2) *For each x, the curve*

$$F_a(x,\cdot):\left[-\frac{(x^2+a^2)^{3/4}}{2},\frac{(x^2+a^2)^{3/4}}{2}\right]\to\{x_3=x\}$$

is a graph.

(P3) *There exists $r_0>0$ so that for every a we have*

$$\left|F_a\left(x,\pm\frac{(x^2+a^2)^{3/4}}{2}\right)-F_a(x,0)\right|>r_0\,.$$

Proof. Since $z_0=0$ and $\phi=dz$, we get (P1) from (1.109). Using $y^2<(x^2+a^2)/4$ on Ω_a, (1.210) and (1.212) give

$$|\partial_y u_a(x,y)|=\frac{2\,|xy|}{(x^2+a^2-y^2)^2+4x^2y^2}\le\frac{4\,|xy|}{(x^2+a^2)^2}\,,\tag{1.216}$$

$$\partial_y v_a(x,y)=\frac{x^2+a^2-y^2}{(x^2+a^2-y^2)^2+4x^2y^2}>\frac{3}{8(x^2+a^2)}\,.\tag{1.217}$$

Set $y_{x,a}=(x^2+a^2)^{3/4}/2$. Integrating (1.216) gives

$$\begin{aligned}\max_{|y|\le y_{x,a}}|u_a(x,y)-u_a(x,0)|&\le\int_0^{y_{x,a}}\frac{4|x|t}{(x^2+a^2)^2}\,dt\\&=\frac{|x|}{2\,(x^2+a^2)^{1/2}}<1\,.\end{aligned}\tag{1.218}$$

Set $\gamma_{x,a}(y) = F_a(x,y)$. Since $v_a(x,0) = 0$ and $\cos(1) > 1/2$, combining (1.207) and (1.218) gives

$$\begin{aligned}\langle \gamma'_{x,a}(y), \gamma'_{x,a}(0)\rangle &= \cosh v_a(x,y)\, \cos(u_a(x,y) - u_a(x,0)) \\ &> \cosh v_a(x,y)/2\,. \end{aligned} \tag{1.219}$$

Here $\gamma'_{x,a}(y) = \partial_y F_a(x,y)$. By (1.219), the angle between $\gamma'_{x,a}(y)$ and $\gamma'_{x,a}(0)$ is always less than $\pi/2$; this gives (P2).

Since $v_a(x,0) = 0$, integrating (1.217) gives

$$\min_{y_{x,a}/2 \leq |y| \leq y_{x,a}} |v_a(x,y)| \geq \int_0^{\frac{y_{x,a}}{2}} \frac{3\,dt}{8(x^2+a^2)} = \frac{3}{32(x^2+a^2)^{1/4}}\,. \tag{1.220}$$

Integrating (1.219) and using (1.220) gives

$$\langle \gamma_{x,a}(y_{x,a}) - \gamma_{x,a}(0), \gamma'_{x,a}(0)\rangle > \frac{(x^2+a^2)^{3/4}}{16}\, \mathrm{e}^{(x^2+a^2)^{-1/4}/11}\,. \tag{1.221}$$

Since $\lim_{s\to 0} s^3\, \mathrm{e}^{s^{-1}/11} = \infty$, (1.221) and its analog for $\gamma_{x,a}(-y_{x,a})$ give (P3). $\square$

Corollary 1.45. *See Figure* 1.30. *Let r_0 be given by (P3).*

(i) *F_a is an embedding.*

(ii) *$F_a(t,0) = (0,0,t)$ for $|t| < 1/2$.*

(iii) *$\{0 < x_1^2 + x_2^2 < r_0^2\} \cap F_a(\Omega_a) = \tilde{\Sigma}_{1,a} \cup \tilde{\Sigma}_{2,a}$ for multi-valued graphs $\tilde{\Sigma}_{1,a}$, $\tilde{\Sigma}_{2,a}$ over $D_{r_0} \setminus \{0\}$.*

Proof. Properties (P1) and (P2) immediately give (i).

Since $z_0 = 0$, $F(0,0) = (0,0,0)$. Integrating (1.206) and using $v_a(x,0) = 0$ then gives (ii).

By (1.115), F_a is "vertical", i.e., $\langle N, (0,0,1)\rangle = 0$, when $|g_a| = 1$. However, $|g_a(x,y)| = 1$ exactly when $y = 0$ so that, by (ii), the image is graphical away from the x_3-axis. Combining this with (P3) gives (iii). $\square$

Corollary 1.45 constructs the embeddings F_a that will be used in Theorem 1.41 and shows property (3). To prove Theorem 1.41, we need therefore only to show (1), (2), and (4).

Proof of Theorem 1.41. By scaling, it suffices to find a sequence $\Sigma_i \subset B_R$ for some $R > 0$. Corollary 1.45 gives minimal embeddings $F_a : \Omega_a \to \mathbb{R}^3$ with $F_a(t,0) = (0,0,t)$ for $|t| < 1/2$ and so (3) holds for any $R \leq r_0$. Set

$$R = \min\{r_0/2, 1/4\}\,, \tag{1.222}$$

$$\Sigma_i = B_R \cap F_{a_i}(\Omega_{a_i})\,, \tag{1.223}$$

where the sequence a_i is to be determined.

To get (1), simply note that, by (1.211),

$$|\,\mathrm{K}_a\,|(0) = a^{-4} \to \infty \text{ as } a \to 0\,. \tag{1.224}$$

We next show (2). First, by (1.211),

$$\sup_a \sup_{\{|x|\geq\delta\}\cap\Omega_a} |\mathrm{K}_a| < \infty \tag{1.225}$$

for all $\delta > 0$. Combined with (3) and Heinz's curvature estimate for minimal graphs (Theorem 2.3 in the next chapter), this gives (2).

To get (4), use Lemma 1.43 to choose $a_i \to 0$ so the mappings F_{a_i} converge uniformly in C^2 on compact subsets to $F_0 : \Omega_0 \to \mathbb{R}^3$. Hence, by Lemma 1.44, $\Sigma_i \setminus \{x_3 = 0\}$ converges to two embedded minimal disks $\Sigma^\pm \subset F_0(\Omega_0^\pm)$ with

$$\Sigma^\pm \setminus \{x_3\text{-axis}\} = \Sigma_1^\pm \cup \Sigma_2^\pm \tag{1.226}$$

for multi-valued graphs $\Sigma_j^\pm$. To complete the proof, we show that each graph $\Sigma_j^\pm$ is ∞-valued (and, hence, spirals into $\{x_3 = 0\}$). Note that, by (3) and (1.207), the level sets $\{x_3 = x\} \cap \Sigma_j^\pm$ are graphs over the line in the direction

$$\lim_{a\to 0} \left(\sin u_a(x,0), -\cos u_a(x,0), 0\right)\,. \tag{1.227}$$

Therefore, since an easy calculation for $0 < t < 1/4$ gives that

$$\lim_{a\to 0} |u_a(t,0) - u_a(2t,0)| = 1/(2t)\,, \tag{1.228}$$

we see that $\{t < |x_3| < 2t\} \cap \Sigma_j^\pm$ contains an embedded N_t-valued graph where

$$N_t \approx \frac{1}{4\pi t} \to \infty \text{ as } t \to 0\,. \tag{1.229}$$

It follows that $\Sigma_j^\pm$ must spiral into $\{x_3 = 0\}$, completing (4). □

Prior to this example, the only local example of a sequence of embedded minimal disks with curvatures blowing up was the sequence of rescaled helicoids where the blow up occurs along a line segment (namely, along the axis of the helicoid). There are now a number of related examples with more complicated blow up sets, including [**De2**], [**HoWh2**], [**Kh**], and [**Kl**].

In each of these local examples, the singular sets where the curvature blows up is a subset of a vertical line. The Colding-Minicozzi limit lamination theorem of [**CM3**]–[**CM6**] implies that the singular set is contained in a Lipschitz curve (see Chapter 8). In [**MeWe**], Meeks and Weber constructed sequences of *bent helicoids* where the surfaces are twisted around a bent curve, just as the helicoid is twisted around its axis. Taking a sequence of these with an increasing number of twists, gives a sequence of embedded minimal surfaces where the curvature blows up along the bent curve. These

surfaces are only embedded in a neighborhood of the curve, where they form double-spiral staircases. This local double-spiral staircase structure turns out to hold in complete generality; this was proven in [**CM3**]–[**CM6**].

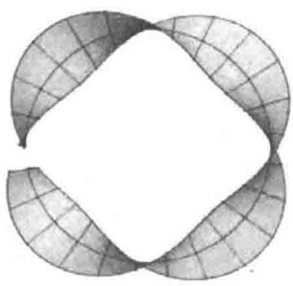

Figure 1.31. A Meeks-Weber bent helicoid with four twists.
Credit: Matthias Weber, www.indiana.edu/~minimal.

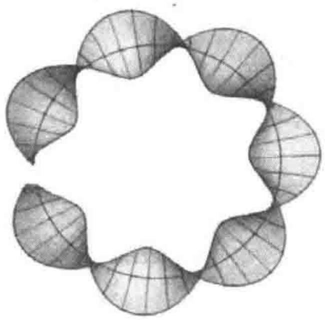

Figure 1.32. A bent helicoid with seven twists.
Credit: Matthias Weber, www.indiana.edu/~minimal.

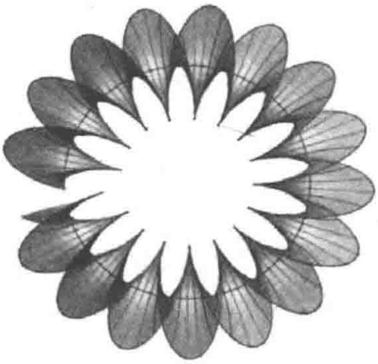

Figure 1.33. As the twists increase, the curvature gets large along a circle.
Credit: Matthias Weber, www.indiana.edu/~minimal.

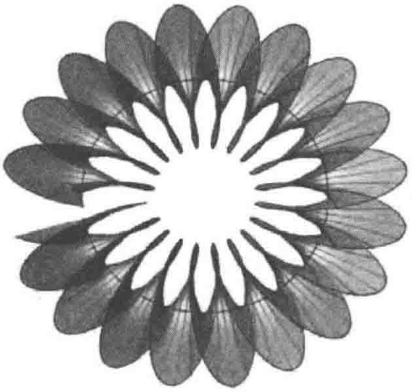

Figure 1.34. The limit will foliate a bent cylinder with the singular circle removed.
Credit: Matthias Weber, www.indiana.edu/~minimal.

Appendix: The Harnack Inequality

We will next recall the Harnack inequality for nonnegative solutions of uniformly elliptic equations. The version that we will use is contained in theorem 8.20 of [**GiTr**] and applies to a very general class of operators. For the next lemma, let $\mathcal{L}$ be a second-order linear differential operator on $\mathbb{R}^n$ given by

$$\mathcal{L}\, u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} + b^i(x)\, u \right) + \sum_{i=1}^{n} c^i(x) \frac{\partial u}{\partial x_i} + d(x)\, u \,, \tag{1.230}$$

where the coefficients a^{ij}, b^i, c^i, d are measurable functions.

Lemma 1.46. *Let $\mathcal{L}$ be a second-order linear differential operator on $\Omega \subset \mathbb{R}^n$ with bounded measurable coefficients a^{ij}, b^i, c^i, d as in* (1.230) *satisfying*

$$\sum_{i,j=1}^{n} a^{ij} x_i x_j \geq \lambda |x|^2 \tag{1.231}$$

for some $\lambda > 0$ and

$$\sum_{i,j=1}^{n} (a^{ij})^2 \leq \Lambda \,, \tag{1.232}$$

$$\lambda^{-2} \sum_{i=1}^{n} (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1} |d(x)| \leq \nu^2 \,, \tag{1.233}$$

for some $\Lambda, \nu < \infty$. Suppose that $u \in C^0(\Omega) \cap W^{1,2}(\Omega)$ satisfies $u \geq 0$ in Ω and $\mathcal{L} u = 0$ weakly in Ω. Then, for any ball $B_{4R}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u \,, \tag{1.234}$$

where $C = C(n, \frac{\Lambda}{\lambda}, \nu R) < \infty$.

By using local coordinates and a covering argument with chains of balls we can extend the Harnack inequality of Lemma 1.46 to elliptic equations on bounded domains in a Riemannian manifold.

Appendix: The Bochner formula

On Euclidean space $\mathbb{R}^n$, partial derivatives commute and, thus,

$$\frac{1}{2} \Delta |\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla \Delta u \rangle \,. \tag{1.235}$$

On a Riemannian manifold M, derivatives do not commute and (1.235) does not hold in general. However, the failure of derivatives to commute is

measured by the Riemann curvature tensor. Using this, Bochner proved the following extremely useful formula:

Proposition 1.47 (Bochner formula). *On a Riemannian manifold M, we have*

$$\frac{1}{2}\,\Delta|\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}_M(\nabla u, \nabla u)\,. \tag{1.236}$$

In the proof, we will use the symmetry of the Hessian Hess_u. The Hessian is the two-tensor given by taking the covariant derivative of the gradient ∇u. Given vector fields V and W, we have

$$\mathrm{Hess}_u(V,W) = (\nabla\nabla u)\ (V,W) = \langle \nabla_V \nabla u, W\rangle\,. \tag{1.237}$$

Using metric compatibility of the connection and the definition of the gradient, we have

$$V(W(u)) = V\langle \nabla u, W\rangle = \langle \nabla_V \nabla u, W\rangle + \langle \nabla u, \nabla_V W\rangle\,, \tag{1.238}$$

so that the bracket $[V,W]$ of V and W applied to u is

$$\begin{aligned}[V,W]u &= V(W(u)) - W(V(u)) \\ &= \mathrm{Hess}_u(V,W) + \langle \nabla u, \nabla_V W\rangle - \mathrm{Hess}_u(W,V) - \langle \nabla u, \nabla_W V\rangle \\ &= \mathrm{Hess}_u(V,W) - \mathrm{Hess}_u(W,V) + [V,W]u\,,\end{aligned} \tag{1.239}$$

where the last equality used the symmetry of the connection (i.e., used that $[V,W] = \nabla_V W - \nabla_W V$).

Proof of Proposition 1.47. Let e_i be a local orthonormal frame for M, so that

$$\Delta v = \sum_i \nabla^2 v(e_i, e_i) = \sum_i \langle \nabla_{e_i} \nabla v, e_i\rangle\,. \tag{1.240}$$

Set $v = \frac{1}{2}\,|\nabla u|^2$. We have

$$\langle \nabla v, e_j\rangle = \langle \nabla_{e_j}\nabla u, \nabla u\rangle = \mathrm{Hess}_u(e_j, \nabla u) = \mathrm{Hess}_u(\nabla u, e_j) = \langle \nabla_{\nabla u}\nabla u, e_j\rangle\,,$$

where the second and fourth equalities are the definition of Hess_u and the third equality is the symmetry of Hess_u. Since this is true for any e_j, we conclude that

$$\nabla v = \nabla_{\nabla u} \nabla u\,. \tag{1.241}$$

Using this in (1.240) gives

$$\begin{aligned}\Delta v &= \sum_i \langle \nabla_{e_i} \nabla_{\nabla u} \nabla u, e_i\rangle \\ &= \sum_i \langle \left(\nabla_{\nabla u}\nabla_{e_i}\nabla u - \nabla_{[\nabla u, e_i]}\nabla u + R_M(\nabla u, e_i)\nabla u\right), e_i\rangle\,,\end{aligned} \tag{1.242}$$

where R_M is the Riemann curvature tensor of M and the second equality is the definition of $R_M(\nabla u, e_i)\nabla u$. The last term in the sum is precisely $\mathrm{Ric}_M(\nabla u, \nabla u)$. The first term can be rewritten as

$$\begin{aligned}\sum_i \langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i\rangle &= \sum_i \nabla u\langle \nabla_{e_i}\nabla u, e_i\rangle - \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle \\ &= \langle \nabla u, \nabla \Delta u\rangle - \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle\,. \end{aligned} \tag{1.243}$$

Using the symmetry of the Hessian of u and then the symmetry of the connection, the second term becomes

$$\begin{aligned} -\sum_i \langle \nabla_{[\nabla u, e_i]}\nabla u, e_i\rangle &= -\sum_i \langle \nabla_{e_i}\nabla u, [\nabla u, e_i]\rangle \\ &= \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{e_i}\nabla u - \nabla_{\nabla u} e_i\rangle \\ &= |\mathrm{Hess}_u|^2 - \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle\,. \end{aligned} \tag{1.244}$$

Putting the formulas for the three terms back into (1.242) gives

$$\begin{aligned} \Delta v = |\mathrm{Hess}_u|^2 &+ \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}_M(\nabla u, \nabla u) \\ &- 2\sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle\,. \end{aligned} \tag{1.245}$$

To complete the proof, we will show that

$$\sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle = 0\,. \tag{1.246}$$

The quickest way to prove (1.246) is to note that this quantity does not depend on the particular choice of frame (since none of the other quantities in (1.245) do) and it is zero in normal coordinates where $\nabla_{e_j} e_i = 0$ at a given point. Since this proof is a little mysterious, so we will give a direct proof of (1.246). To do this, expand $\nabla_{e_i}\nabla u$ and $\nabla_{\nabla u} e_i$ in the basis as

$$\nabla_{e_i}\nabla u = \sum_j a_{ij}\, e_j\,, \tag{1.247}$$

$$\nabla_{\nabla u} e_i = \sum_j b_{ij}\, e_j\,, \tag{1.248}$$

so that the left-hand side of (1.246) is just $\sum_{i,j}(a_{ij}\, b_{ij})$. The proof of (1.246) now follows from three facts:

(1) $a_{ij} = \langle \nabla_{e_i}\nabla u, e_j\rangle$ is symmetric, i.e., $a_{ij} = a_{ji}$.
(2) $b_{ij} = \langle \nabla_{\nabla u} e_i, e_j\rangle$ is skew-symmetric, i.e., $b_{ij} = -b_{ji}$.
(3) If a_{ij} is symmetric and b_{ij} is skew-symmetric, then $\sum_{i,j}(a_{ij}\, b_{ij}) = 0$.

Fact (1) is just symmetry of the Hessian of u. Fact (2) follows from metric compatibility since $\nabla u \langle e_i, e_j \rangle = 0$. To prove (3), note that

$$\sum_{i,j} (a_{ij}\, b_{ij}) = \sum_{i,j} (a_{ji}\, b_{ji}) = -\sum_{i,j} (a_{ij}\, b_{ij}) \,, \tag{1.249}$$

so this term vanishes, completing the proof. □

Chapter 2

Curvature Estimates and Consequences

In this chapter, we will give various generalizations of the Bernstein theorem discussed in Chapter 1. We begin by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we will discuss various applications of such an inequality. Our first application is to a theorem of Choi and Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding and Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen, Simon and Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the norm squared of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We introduce a notion of "almost-stabilility" that plays a crucial role in understanding embedded surfaces. Next, we return to multi-valued minimal graphs and prove an important result from [**CM3**] which states that the separation grows sublinearly if the multi-valued graph has enough sheets. We close the chapter with a discussion of minimal cones in Euclidean space and the counterexample of Bombieri, De Giorgi and Giusti to the Bernstein theorem in dimension greater than seven.

1. Simons' Inequality

In this section, we will derive a very useful differential inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections of this chapter, we will discuss various applications of such an inequality. This inequality, originally due to J. Simons [**Sim**], asserts in its most general form that for a minimal hypersurface $\Sigma^{n-1} \subset M^n$,

$$\Delta_\Sigma |A|^2 \geq -C\,(1+|A|^2)^2\,, \tag{2.1}$$

where C depends on the curvature of M and its covariant derivative.

We will now derive Simons' inequality in its original form, see [**ScSiYa**] for the more general inequality described above.

Lemma 2.1 (Simons' Inequality, [**Sim**]). *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is a minimal hypersurface; then*

$$\Delta_\Sigma |A|^2 \geq -2\,|A|^4 + 2\left(1+\frac{2}{n-1}\right)|\nabla_\Sigma |A||^2\,. \tag{2.2}$$

Note that this can equivalently be expressed as

$$|A|\,\Delta_\Sigma\,|A| + |A|^4 \geq \frac{2}{n-1}\,|\nabla_\Sigma |A||^2\,. \tag{2.3}$$

Before proving the lemma, we briefly recall the definition of the Riemann curvature R and the Gauss equation which expresses R in terms of the second fundamental form. Namely, if E_i is an orthonormal frame for a manifold Σ with metric g and connection ∇^Σ, then the curvature R is given by

$$\begin{aligned} R_{ijkl} &= g(R(E_i,E_j)E_k,E_\ell) \\ &= g(\nabla^\Sigma_{E_j}\nabla^\Sigma_{E_i}E_k - \nabla^\Sigma_{E_i}\nabla^\Sigma_{E_j}E_k + \nabla^\Sigma_{\nabla^\Sigma_{E_i}E_j - \nabla^\Sigma_{E_j}E_i}E_k,\, E_\ell)\,. \end{aligned} \tag{2.4}$$

When Σ is a hypersurface in $\mathbb{R}^n$ with the induced metric and connection, then the Gauss equation expresses the curvature R in terms of the second fundamental form of Σ:

$$\mathrm{R}_{ijk\ell} = a_{ik}\,a_{j\ell} - a_{jk}\,a_{i\ell}\,, \tag{2.5}$$

where $a_{jk} = \langle A(E_j,E_k),E_n\rangle$ and E_n is a unit normal.

Proof of Lemma 2.1. Let E_i for $i = 1, \dots, n$ be a locally defined orthonormal frame in a neighborhood of some $x \in \Sigma$ such that E_n is normal to Σ. Let a be the symmetric two-tensor on Σ given by

$$a(X,Y) = \langle A(X,Y), E_n \rangle = \langle \nabla_X Y, E_n \rangle = -\langle \nabla_X E_n, Y \rangle \,, \tag{2.6}$$

and set $a_{ij} = -\langle \nabla_{E_i} E_n, E_j \rangle$. That is,

$$a = \sum_{i,j=1}^{n-1} a_{ij}\, \Theta_i \otimes \Theta_j \,, \tag{2.7}$$

where Θ_i is the dual (to E_i) orthonormal frame. Let $a_{..,k}$ and $a_{ij,k}$ be defined by

$$a_{..,k} = \sum_{i,j=1}^{n-1} a_{ij,k}\, \Theta_i \otimes \Theta_j = \nabla_{E_k} a \,. \tag{2.8}$$

Since a is a symmetric two-tensor, it follows that $a_{..,k}$ is also a symmetric two-tensor. Since the curvature R of $\mathbb{R}^n$ vanishes, we get the *Codazzi equation*:

$$\begin{aligned} a_{ij,k} &= \left(\nabla^T_{E_k} a\right)(E_i, E_j) \\ &= E_k\, a(E_i, E_j) - a(\nabla^T_{E_k} E_i, E_j) - a(E_i, \nabla^T_{E_k} E_j) \\ &= -E_k \langle \nabla_{E_i} E_n, E_j \rangle + \langle \nabla_{\nabla^T_{E_k} E_i} E_n, E_j \rangle + \langle \nabla_{E_i} E_n, \nabla_{E_k} E_j \rangle \\ &= -\langle \nabla_{E_k} \nabla_{E_i} E_n, E_j \rangle + \langle \nabla_{\nabla^T_{E_k} E_i} E_n, E_j \rangle \\ &= -\langle \nabla_{E_i} \nabla_{E_k} E_n, E_j \rangle + \langle \nabla_{\nabla^T_{E_i} E_k} E_n, E_j \rangle = a_{kj,i} \,. \end{aligned} \tag{2.9}$$

Note that the second equality above is just the Leibniz rule, the third is the definition of A, and the last uses that R_{iknj} vanishes on $\mathbb{R}^n$. Therefore, let $a_{..,.}$ be the symmetric three-tensor given by

$$a_{..,.} = \sum_{i,j,k=1}^{n-1} a_{ij,k}\, \Theta_i \otimes \Theta_j \otimes \Theta_k \,. \tag{2.10}$$

Next, we define a symmetric three-tensor $a_{..,.\ell}$ by

$$a_{..,.\ell} = \sum_{i,j,k=1}^{n-1} a_{ij,k\ell}\, \Theta_i \otimes \Theta_j \otimes \Theta_k = \nabla_{E_\ell} a_{..,.} \,. \tag{2.11}$$

Using the Leibniz rule, we compute that

$$
\begin{aligned}
a_{ij,k\ell} &= \nabla^T_{E_\ell}\, a_{\cdot,\cdot,\cdot}(E_i, E_j, E_k) \\
&= E_\ell\, a_{\cdot,\cdot,\cdot}(E_i, E_j, E_k) - a_{\cdot,\cdot,\cdot}(\nabla^T_{E_\ell} E_i, E_j, E_k) \\
&\quad - a_{\cdot,\cdot,\cdot}(E_i, \nabla^T_{E_\ell} E_j, E_k) - a_{\cdot,\cdot,\cdot}(E_i, E_j, \nabla^T_{E_\ell} E_k) \\
&= E_\ell\, a_{\cdot,\cdot,k}(E_i, E_j) - a_{\cdot,\cdot,k}(\nabla^T_{E_\ell} E_i, E_j) \\
&\quad - a_{\cdot,\cdot,k}(E_i, \nabla^T_{E_\ell} E_j) - \nabla^T_{\nabla^T_{E_\ell} E_k} a(E_i, E_j)\,.
\end{aligned} \tag{2.12}
$$

Applying the Leibniz rule again then gives

$$
\begin{aligned}
a_{ij,k\ell} &= E_\ell\, E_k\, a(E_i, E_j) - E_\ell\, a(\nabla^T_{E_k} E_i, E_j) - E_\ell\, a(E_i, \nabla^T_{E_k} E_j) \\
&\quad - E_k\, a(\nabla^T_{E_\ell} E_i, E_j) + a(\nabla^T_{E_k} \nabla^T_{E_\ell} E_i, E_j) + a(\nabla^T_{E_\ell} E_i, \nabla^T_{E_k} E_j) \\
&\quad - E_k\, a(E_i, \nabla^T_{E_\ell} E_j) + a(\nabla^T_{E_k} E_i, \nabla^T_{E_\ell} E_j) + a(E_i, \nabla^T_{E_k} \nabla^T_{E_\ell} E_j) \\
&\quad - \nabla^T_{E_\ell} E_k\, a(E_i, E_j) + a(\nabla^T_{\nabla^T_{E_\ell} E_k} E_i, E_j) + a(E_i, \nabla^T_{\nabla^T_{E_\ell} E_k} E_j) \\
&= a_{ij,\ell k} + \sum_{m=1}^{n-1} \mathrm{R}_{\ell k i m}\, a_{mj} + \sum_{m=1}^{n-1} \mathrm{R}_{\ell k j m}\, a_{mi}\,.
\end{aligned} \tag{2.13}
$$

(To see the last equality, notice that most of the terms come in obvious symmetric pairs — 1 and 10, 2 and 4, 3 and 7, 6 and 8 — and the remaining terms — 5 and 11, 9 and 12 — give the curvature terms.) Note that this computation holds for any symmetric two-tensor a.

Using the Gauss equations (2.5), we can rewrite (2.13) as

$$
\begin{aligned}
a_{ik,jk} &= a_{ik,kj} + \sum_{m=1}^{n-1} \mathrm{R}_{kjim}\, a_{mk} + \sum_{m=1}^{n-1} \mathrm{R}_{kjkm}\, a_{mi} \\
&= a_{ik,kj} + \sum_{m=1}^{n-1} (a_{ki}\, a_{jm} - a_{ji}\, a_{km})\, a_{mk} \\
&\quad + \sum_{m=1}^{n-1} (a_{kk}\, a_{jm} - a_{jk}\, a_{km})\, a_{mi}\,.
\end{aligned}
\tag{2.14}
$$

Finally, we are ready to show Simons' equation. Using (2.14), we get that

(2.15)

$$
\begin{aligned}
\frac{1}{2}\,\Delta_\Sigma\,|A|^2 &= \sum_{i,j=1}^{n-1} a_{ij}\,\Delta_\Sigma a_{ij} + \sum_{i,j=1}^{n-1} |\nabla_\Sigma a_{ij}|^2 \\
&= \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{ij,kk} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 = \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{ik,jk} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
&= \sum_{i,j,k=1}^{n-1} a_{ij}\, a_{kk,ij} + \sum_{i,j,k,m=1}^{n-1} a_{ij}\,(a_{ki}\, a_{jm} - a_{ji}\, a_{km})\, a_{mk} \\
&\quad + \sum_{i,j,k,m=1}^{n-1} a_{ij}\,(a_{kk}\, a_{jm} - a_{jk}\, a_{km})\, a_{mi} + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2 \\
&= - \sum_{i,j,k,m=1}^{n-1} a_{ij}^2\, a_{km}^2 + \sum_{i,j,k=1}^{n-1} a_{ij,k}^2\,.
\end{aligned}
$$

Note that the last equality used that $\sum_k a_{kk} = 0$ (i.e., that Σ is minimal) and used the symmetry of a_{ij} to cancel two of the sums. Therefore, we get

$$
\Delta_\Sigma |A|^2 = -2\,|A|^4 + 2 \sum_{i,j,k=1}^{n-1} a_{ij,k}^2
\tag{2.16}
$$

which is *Simons' equation.*

We will now see how to obtain Simons' inequality from this. Observe first that, since a is symmetric, we may choose E_i, $i = 1, \ldots, n-1$, such that at x we have

$$
a_{ij} = \lambda_i\, \delta_{ij}\,.
\tag{2.17}
$$

Since $\nabla|A|^2 = 2\,|A|\,\nabla|A|$, we have

$$
\begin{aligned}
(2.18) \qquad 4\,|A|^2\,|\nabla|A||^2 = |\nabla|A|^2|^2 &= \sum_{k=1}^{n-1}[(\sum_{i,j=1}^{n-1} a_{ij}^2)_k]^2 \\
&= 4\sum_{k=1}^{n-1}(\sum_{i=1}^{n-1} a_{ii,k}\,\lambda_i)^2 \le 4\,|A|^2 \sum_{i,k=1}^{n-1} a_{ii,k}^2\,,
\end{aligned}
$$

where the third equality used (2.17) and the inequality used that the inner product of the vectors $(a_{11,k}, \dots, a_{n-1,n-1,k})$ and $(\lambda_1, \dots, \lambda_{n-1})$ is bounded by the product of their lengths. Hence (2.18) gives

$$
(2.19) \qquad |\nabla|A||^2 \le \sum_{i,k=1}^{n-1} a_{ii,k}^2\,.
$$

Therefore, by minimality and (2.19),

$$
\begin{aligned}
|\nabla|A||^2 &\le \sum_{i,k=1}^{n-1} a_{ii,k}^2 = \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1} a_{ii,i}^2 \\
(2.20) \qquad &= \sum_{i\neq k} a_{ii,k}^2 + \sum_{i=1}^{n-1}\left(\sum_{i\neq j} a_{jj,i}\right)^2 \\
&\le \sum_{i\neq k} a_{ii,k}^2 + (n-2)\sum_{i=1}^{n-1}\sum_{i\neq j} a_{jj,i}^2 = (n-1)\sum_{i\neq k} a_{ii,k}^2 \\
&= (n-1)\sum_{i\neq k} a_{ik,i}^2 = \frac{n-1}{2}\left(\sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2\right).
\end{aligned}
$$

The second inequality in (2.20) used the algebraic fact that

$$
(2.21) \qquad \left(\sum_{j=1}^{n-2} b_j\right)^2 \le (n-2)\sum_{j=1}^{n-2} b_j^2\,.
$$

From (2.19) and (2.20), we get

$$
\begin{aligned}
(1+\frac{2}{n-1})\,|\nabla|A||^2 &\le \sum_{i,k=1}^{n-1} a_{ii,k}^2 + \sum_{i\neq k} a_{ik,i}^2 + \sum_{i\neq k} a_{ki,i}^2 \\
(2.22) \qquad &\le \sum_{i,j,k=1}^{n-1} a_{ij,k}^2\,.
\end{aligned}
$$

Combining (2.22) with Simons' equation (i.e., (2.16)) yields Simons' inequality. □

1.1. Simons' equation for surfaces. The following is a consequence of this lemma:

If $n = 3$ and $\Sigma^2 \subset \mathbb{R}^3$ is a minimal surface, then

$$\Delta \log |A|^2 = \frac{\Delta |A|^2 - 4\,|\nabla |A||^2}{|A|^2} = -2\,|A|^2 \,. \tag{2.23}$$

This follows since, for a surface, Simons' equation actually implies the equation

$$\Delta |A|^2 = -2\,|A|^4 + 4\,|\nabla |A||^2 \tag{2.24}$$

and not just an inequality. This is easily seen since, in the case of a surface, the Cauchy-Schwarz inequalities applied above are equalities by minimality.

Equation (2.23) has the nice geometric interpretation that

$$|A|\,\langle \cdot, \cdot \rangle \tag{2.25}$$

is a flat (possibly singular) metric on Σ. Namely, if (Σ, g) is a Riemann surface and f is a positive function on Σ, then the curvature of $(\Sigma, f^2\, g)$ is given by

$$\mathrm{K}_g = \Delta_g \log f + f^2\, \mathrm{K}_{f^2\, g} \,. \tag{2.26}$$

Taking $f = |A|^{\frac{1}{2}}$ in this formula, where $|A| > 0$, and using Simons' equation in dimension two yields, by minimality,

$$\begin{aligned} -|A|^2 = 2\mathrm{K}_\Sigma &= \Delta_\Sigma \log |A| + 2\,|A|\,\mathrm{K}_{|A|\,\Sigma} \\ &= -|A|^2 + 2\,|A|\,\mathrm{K}_{|A|\,\Sigma} \,. \end{aligned} \tag{2.27}$$

This clearly implies that $|A|\,\langle \cdot, \cdot \rangle$ is a flat metric on Σ where $|A| > 0$.

The formula for the curvature of the conformally changed metric (i.e., formula (2.26)) can easily be proven using moving frames. Namely, for $x \in \Sigma$, let E_1 and E_2 be an orthonormal frame of (Σ, g) in a neighborhood x and let Θ_i be the dual orthonormal coframe. Then $\bar{\Theta}_i = f\,\Theta_i$ is an orthonormal coframe for $(\Sigma, f^2\, g)$. Further from the Cartan equations we see that

$$\bar{\omega}_{1,2} = \omega_{1,2} - E_2(\log f)\,\Theta_1 + E_1(\log f)\,\Theta_2 \tag{2.28}$$

and therefore

$$\bar{\Omega}_{1,2} = d\bar{\omega}_{1,2} = d\omega_{1,2} + [E_1(E_1(\log f)) + E_2(E_2(\log f))]\,\Theta_1 \wedge \Theta_2$$

$$(2.29) \qquad - E_2(\log f)\, d\Theta_1 + E_1(\log f)\, d\Theta_2$$

$$= \Omega_{1,2} + \frac{\Delta_g \log f}{f^2}\,\bar{\Theta}_1 \wedge \bar{\Theta}_2\,,$$

and the claim follows.

2. Small Energy Curvature Estimates for Minimal Surfaces

In this section, we will obtain a priori interior curvature bounds for minimal surfaces which have small total curvature. This will be our basic curvature estimate and it will be used in proving other more general curvature estimates, e.g., for minimal graphs, stable minimal surfaces, minimal surfaces with small area, etc.

We will next prove the "small total curvature" estimate of Choi and Schoen [**CiSc**]. This is one example of a type of estimate that has been quite important in geometric analysis. It should be compared with similar estimates of J. Sacks and K. Uhlenbeck [**SaUh**] (for harmonic maps), M. T. Anderson [**An**], and B. White [**Wh1**].

Theorem 2.2 (Choi-Schoen, [**CiSc**]). *There exist $\epsilon, \rho > 0$ (depending on M) so that if $r_0 < \rho$, $\Sigma^2 \subset M$ is a compact minimal surface with $\partial\Sigma \subset \partial B_{r_0}(x)$, $0 < \delta \leq 1$, and*

$$\int_{B_{r_0} \cap \Sigma} |A|^2 < \delta\,\epsilon\,, \tag{2.30}$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0 - \sigma}(x)$,

$$\sigma^2\,|A|^2(y) \leq \delta\,. \tag{2.31}$$

Proof. We shall give the proof for the case $M^n = \mathbb{R}^n$ and leave the necessary modifications for the general case to the reader; cf. Chapter 7.

Set $F = (r_0 - r)^2\,|A|^2$ on $B_{r_0} \cap \Sigma$ and observe that F vanishes on ∂B_{r_0}. Let $x_0 \in \Sigma$ be a point where F achieves its maximum and note that it is enough to show that $F(x_0) < \delta$. We will show that $F(x_0) \geq \delta$ leads to a contradiction for $\epsilon > 0$ sufficiently small.

Define $\sigma > 0$ by

$$\sigma^2\,|A|^2(x_0) = \frac{\delta}{4}\,. \tag{2.32}$$

In particular, since $F(x_0) \geq \delta$, we see that

$$2\,\sigma \leq r_0 - r(x_0)\,. \tag{2.33}$$

Consequently, by the triangle inequality, we have on $B_\sigma(x_0)$ that

$$\frac{1}{2} \leq \frac{r_0 - r}{r_0 - r(x_0)} \leq 2\,.$$

Since F achieves its maximum at x_0, we conclude that

$$\begin{aligned}(r_0 - r(x_0))^2 \sup_{B_\sigma(x_0)\cap\Sigma} |A|^2 &\leq 4 \sup_{B_\sigma(x_0)\cap\Sigma} F(x) = 4\,F(x_0)\\ &= 4\,(r_0 - r(x_0))^2\,|A|^2(x_0)\,.\end{aligned} \tag{2.34}$$

Dividing through by $(r_0 - r(x_0))^2$ and using the definition of σ gives

$$\sup_{B_\sigma(x_0)\cap\Sigma} |A|^2 \leq 4\,|A|^2(x_0) = \delta\,\sigma^{-2}\,. \tag{2.35}$$

After rescaling $B_\sigma(x_0) \cap \Sigma$ to unit size (and still calling it Σ!), we have

$$\sup_{B_1(x_0)\cap\Sigma} |A|^2 \leq 4\,|A|^2(x_0) = \delta \leq 1\,. \tag{2.36}$$

By Simons' inequality on $B_1(x_0) \cap \Sigma$,

$$\Delta|A|^2 \geq -2\,|A|^2\,. \tag{2.37}$$

The desired contradiction now follows from the mean value inequality. Namely, Corollary 1.16 implies that

$$\frac{\delta}{4} = |A|^2(x_0) \leq \mathrm{e}\,\frac{\int_{B_1(x_0)} |A|^2}{\pi} < \frac{\mathrm{e}}{\pi}\,\delta\,\epsilon\,, \tag{2.38}$$

which is a contradiction provided that ϵ is chosen sufficiently small. □

2.1. Heinz's curvature estimate for minimal graphs. As an immediate application, we next prove E. Heinz's curvature estimate for minimal graphs. For this, let D_{r_0} be a disk in $\mathbb{R}^2$ of radius r_0.

Theorem 2.3 (Heinz, [**He**]). *If $u : D_{r_0} \to \mathbb{R}$ satisfies the minimal surface equation, then for $\Sigma = \mathrm{Graph}_u$ and $0 < \sigma \leq r_0$,*

$$\sigma^2 \sup_{D_{r_0-\sigma}} |A|^2 \leq C\,. \tag{2.39}$$

Proof. Clearly, it suffices to prove the case where $\sigma = r_0$. We have already shown in Corollary 1.20 of the previous chapter that, for any $\kappa > 1$,

$$\int_{B_{r_0/\sqrt{\kappa}}\cap\mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log\kappa}\,. \tag{2.40}$$

We choose $\kappa > 1$ so that

$$\frac{C}{\log\kappa} < \epsilon\,, \tag{2.41}$$

where ϵ is given by Theorem 2.2 and the theorem now easily follows. $\square$

So far we have seen that both minimal surfaces with small total curvature and minimal graphs satisfy a priori curvature estimates. We will see below that small area also implies a curvature estimate (see Theorem 2.15).

2.2. Estimates for intrinsic balls. For this, it will be convenient to let $B_r^\Sigma(x)$ denote the intrinsic ball in Σ of radius r centered at x:

$$B_r^\Sigma(x) = \{y \in \Sigma \mid \operatorname{dist}_\Sigma(x,y) < r\}\,. \tag{2.42}$$

Obviously, we always have that $B_r^\Sigma(x) \subset B_r(x)$. We will use this notation for the remainder of this section.

We will use the following elementary lemma which shows that small intrinsic balls in Σ are graphical when there is a curvature bound.

Lemma 2.4 (Small Curvature Implies Graphical). *Let $\Sigma^2 \subset \mathbb{R}^n$ be an immersed surface with*

$$16\, s^2 \sup_\Sigma |A_\Sigma|^2 \le 1\,. \tag{2.43}$$

If $x \in \Sigma$ and $\operatorname{dist}_\Sigma(x, \partial\Sigma) \ge 2\,s$, then

(1) *$B_{2s}^\Sigma(x)$ can be written as a graph of a function u over $T_x\Sigma$ with $|\nabla u| \le 1$ and $\sqrt{2}\, s\, |\operatorname{Hess}_u| \le 1$;*
(2) *The connected component of $B_s(x) \cap \Sigma$ containing x is contained in $B_{2s}^\Sigma(x)$.*

Part (2) of the lemma is a chord-arc bound that relates intrinsic and extrinsic distances.

Proof of Lemma 2.4. Within this lemma, we define

$$d_{x,y} \equiv \operatorname{dist}_{\mathbf{S}^{n-1}}(N(x), N(y))\,. \tag{2.44}$$

Recall that $|\nabla N| \le |A|$ by (1.84). Therefore, given $y \in \Sigma$, integrating (2.43) along the geodesic joining x and y gives

$$\sup_{y \in B_{2s}^\Sigma(x)} d_{x,y} \le \frac{1}{2} < \frac{\pi}{4}\,. \tag{2.45}$$

Since $d_{x,y} < \pi/2$ on this set, it follows that $B_{2s}^\Sigma(x)$ is contained in the graph of a function u over (a subset of) $T_x\Sigma$. Moreover, using (1.3), we have that

$$1 + |\nabla u|^2 = \langle N(x), N(y)\rangle^{-2} = \cos^{-2} d_{x,y}\,, \tag{2.46}$$

so that (2.45) implies $|\nabla u| \le 1$. The Hessian estimate on u comes from the gradient and curvature bounds together with (1.94):

$$|\operatorname{Hess}_u|^2 \le (1 + |\nabla u|^2)^2\, |A|^3 \le 8 \frac{1}{16\, s^2}\,. \tag{2.47}$$

Finally, to show that the component of $B_s(x) \cap \Sigma$ containing x is contained in $B^{\Sigma}_{2s}(x)$, we will show that $\partial B^{\Sigma}_{2s}(x)$ lies outside of $\overline{B_s(x)}$. By elementary geometry, we know that each point on $\partial B^{\Sigma}_{2s}(x)$ is the endpoint of an intrinsic geodesic in Σ that starts at x and has length $2s$. Let

$$\gamma : [0, 2s] \to \Sigma$$

be one such geodesic parametrized by arclength. Observe that

$$\left|\partial_t \langle \gamma'(t), \gamma'(0)\rangle\right| \leq |A|(\gamma(t)) \leq 1/4s\,. \tag{2.48}$$

Since $\langle \gamma'(0), \gamma'(0)\rangle = 1$, integrating this from 0 to t for any $t \leq 2s$ gives

$$\langle \gamma'(t), \gamma'(0)\rangle \geq 1 - \frac{t}{4s} \geq 1/2\,. \tag{2.49}$$

Integrating this from 0 to $2s$ gives

$$\langle [\gamma(t) - \gamma(0)]\,, \gamma'(0)\rangle \geq \int_0^{2s} \left(1 - \frac{t}{4s}\right)\, dt = \frac{3s}{2}\,. \tag{2.50}$$

It follows that $\gamma(2s)$ lies outside of $B_{3s/2}(x)$. Since this holds for any such γ, we get that

$$B_{3s/2}(x) \cap \partial B^{\Sigma}_{2s}(x) = \emptyset\,, \tag{2.51}$$

which completes the proof. □

Note that, if we replace the bound 1 on the right-hand side of (2.43) by some $\delta < 1$, then we would get that

$$\sup_{y \in \tilde{\Sigma}} d_{x,y} \leq \frac{\delta}{2}\,. \tag{2.52}$$

If $0 < s < \frac{\pi}{4}$, then

$$\frac{\partial}{\partial s} \cos^{-2} s = 2\, \sin s\, \cos^{-3} s \leq 4\,, \tag{2.53}$$

so that $\cos^{-2} s \leq 1 + 4\, s$. Therefore, (2.46) and (2.52) imply that

$$|\nabla u|^2 \leq 4\, \delta\,. \tag{2.54}$$

We can now see that:

Remark 2.5. *Theorem 2.2 holds with intrinsic balls instead of extrinsic balls (with slightly different constants).*

To see this, let r denote the intrinsic distance instead, replace extrinsic balls with intrinsic balls, and follow the argument verbatim up to (2.37). At this point, arguing as in Lemma 2.4, (2.36) implies that $B^{\Sigma}_1(x_0)$ contains the connected component of $B_{1/2}(x_0)$ which contains x_0. We can now replace $B_1(x_0)$ with $B_{1/2}(x_0)$ and apply the mean value inequality to complete the proof as before.

2.3. Estimates on higher derivatives. By Lemma 2.4, the curvature bounds of the previous result, (2.31) imply that the connected components of $B_{\sigma/4}(y) \cap \Sigma$ are graphical with bounded gradient and Hessian. This implies that the associated minimal surface equation is a uniformly elliptic divergence form equation with C^1 coefficients. Standard elliptic theory then yields uniform estimates (see, for instance, corollary 16.7 of [**GiTr**]). This will be used later in Chapter 7 when we study compactness theorems for minimal surfaces.

3. Curvature and Area

In this section, we will prove a number of two-dimensional results that rely on estimates for area and total curvature. The basic ideas are simple, but are still quite useful. The results from this section are taken from the papers [**CM2**] and [**CM4**].

The starting point is the next lemma relating area growth and total curvature:

Lemma 2.6. *If $B^{\Sigma}_{r_0}(x) \subset \Sigma^2$ is disjoint from the cut locus of x, then*

$$\text{Length}(\partial B^{\Sigma}_{r_0}) - 2\pi r_0 = - \int_0^{r_0} \int_{B^{\Sigma}_{\rho}} \mathrm{K}_{\Sigma} \, , \tag{2.55}$$

$$\text{Area}(B^{\Sigma}_{r_0}(x)) - \pi \, r_0^2 = - \int_0^{r_0} \int_0^{\tau} \int_{B^{\Sigma}_{\rho}(x)} \mathrm{K}_{\Sigma} \, . \tag{2.56}$$

Proof. For $0 < t \leq r_0$, by the Gauss-Bonnet theorem (see page 274 of [**dC1**]),

$$\frac{d}{dt} \int_{\partial B^{\Sigma}_t} 1 = \int_{\partial B^{\Sigma}_t} k_g = 2\,\pi - \int_{B^{\Sigma}_t} \mathrm{K}_{\Sigma} \, , \tag{2.57}$$

where k_g is the geodesic curvature of ∂B^{Σ}_t. Integrating (2.57) gives the lemma. □

The next corollary from [**CM4**] specializes Lemma 2.6 to the case where Σ is minimal.

Corollary 2.7. *If $\Sigma^2 \subset \mathbb{R}^3$ is immersed and minimal, $B^{\Sigma}_{r_0} \subset \Sigma^2$ is a disk, and $B^{\Sigma}_{r_0} \cap \partial \Sigma = \emptyset$, then*

$$\begin{aligned} t^2 \int_{B^{\Sigma}_{r_0 - 2t}} |A|^2 &\leq \int_{B^{\Sigma}_{r_0}} |A|^2 \, (r_0 - r)^2/2 = \int_0^{r_0} \int_0^{\tau} \int_{B^{\Sigma}_{\rho}(x)} |A|^2 \\ &= 2 \, (\text{Area} \, (B^{\Sigma}_{r_0}) - \pi \, {r_0}^2) \leq r_0 \, \text{Length}(\partial B^{\Sigma}_{r_0}) - 2\pi \, r_0^2 \, . \end{aligned} \tag{2.58}$$

Proof. Since Σ is minimal, the Gauss equation gives $|A|^2 = -2\,\mathrm{K}_\Sigma$ and hence, by Lemma 2.6, we get

$$t^2 \int_{B^\Sigma_{r_0-2t}} |A|^2 \leq t \int_0^{r_0-t} \int_{B^\Sigma_\rho} |A|^2 \leq \int_0^{r_0} \int_0^\tau \int_{B^\Sigma_\rho} |A|^2$$
$$= 2\,(\mathrm{Area}\,(B^\Sigma_{r_0}) - \pi\, {r_0}^2)\,. \tag{2.59}$$

The first equality in (2.58) follows from the coarea formula and integration by parts twice:

$$\int_0^{r_0} f(t)\, g''(t)\, dt = \int_0^{r_0} f''(t)\, g(t)\, dt\,, \tag{2.60}$$

with $f(t) = \int_0^t \int_{B^\Sigma_s} |A|^2$ and $g(t) = (r_0 - t)^2/2)$.

To get the last inequality in (2.58), note that $\frac{d^2}{dt^2}\mathrm{Length}(\partial B^\Sigma_t) \geq 0$ by (2.57) hence

$$t\,\frac{d}{dt}\mathrm{Length}(\partial B^\Sigma_t) \geq \mathrm{Length}(\partial B^\Sigma_t)$$

and, consequently,

$$\frac{d}{dt}\left(\mathrm{Length}(\partial B^\Sigma_t)/t\right) \geq 0\,.$$

From this it follows easily. □

3.1. Estimates for stable minimal surfaces. For stable minimal surfaces in three-manifolds, no assumption on the total curvature, area, or topology is needed to get curvature estimates. This is not known in higher dimensions. We will limit the discussion to surfaces in $\mathbb{R}^3$ for simplicity and clarity.

The starting point is a uniform area bound for 2-sided stable minimal surfaces in $\mathbb{R}^3$:

Theorem 2.8 (Colding-Minicozzi [**CM2**]). *If $\Sigma^2 \subset \mathbb{R}^3$ is stable and 2-sided and $B^\Sigma_{r_0}$ is simply connected, then*

$$\mathrm{Area}\,(B^\Sigma_{r_0}) \leq \frac{4\pi}{3}\, r_0^2\,. \tag{2.61}$$

Proof. Let r be the intrinsic distance function to the center of $B^\Sigma_{r_0}$. Combining Corollary 2.7 and then the stability inequality (Lemma 1.32) gives

$$4\,(\mathrm{Area}\,(B^\Sigma_{r_0}) - \pi r_0^2) = \int_{B^\Sigma_{r_0}} |A|^2\,(r_0 - r)^2 \leq \int_{B^\Sigma_{r_0}} |\nabla r|^2$$
$$= \mathrm{Area}\,(B^\Sigma_{r_0})\,, \tag{2.62}$$

where the last equality used that $|\nabla r| = 1$. The theorem follows. □

As an immediate consequence, we can let $r_0 \to \infty$ in Theorem 2.8 to obtain Bernstein-type theorems (see [**FiSc**] or [**dCP**]):

Corollary 2.9. *If $\Sigma^2 \subset \mathbb{R}^3$ is complete, stable and 2-sided, then it is flat.*

Proof. By Proposition 1.39, the universal cover of Σ is also stable, so we may assume that Σ is simply connected. Since Σ has nonpositive curvature (by the Gauss equation), geodesic balls in Σ are simply connected and, thus, Theorem 2.8 implies that Σ has quadratic area growth. Proposition 1.37 then implies that Σ is parabolic. Finally, Corollary 1.40 implies that Σ is flat. □

Following [**CM2**], we can use Theorem 2.8 to prove the following estimate of R. Schoen:

Theorem 2.10 ([**Sc1**]). *If $\Sigma^2 \subset M^3$ is an immersed stable minimal surface with trivial normal bundle and $B^\Sigma_{r_0} = B^\Sigma_{r_0}(x) \subset \Sigma \setminus \partial\Sigma$, where $|K_M| \leq k^2$ and $r_0 < \rho_1(\pi/k, k)$, then for some $C = C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B^\Sigma_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.63}$$

Proof. We will give the proof from [**CM2**] for $M = \mathbb{R}^3$ and leave the easy modifications to the reader. We begin with two observations. The first observation is that it suffices to prove (2.63) with $\sigma = r_0$, i.e., to prove that

$$|A|^2(x) \leq C\,r_0^{-2}\,. \tag{2.64}$$

Second, we may as well assume that Σ is a topological disk since Proposition 1.39 implies that the universal cover of Σ is also stable.

Since Σ has nonpositive curvature and we can assume that it is a disk, Theorem 2.8 applies to give $\mathrm{Area}\,(B^\Sigma_s(x)) \leq \frac{4\pi}{3}\,s^2$ for every $s \leq r_0$. This quadratic area bound allows us to use the logarithmic cutoff trick to get small total curvature on a sub-ball. Namely, let r denote the distance in Σ to x, fix a large integer $n > 0$ to be chosen, and define η by

$$\eta = \begin{cases} 1 & \text{if } r \leq \mathrm{e}^{-n}\,r_0\,, \\ \frac{\log r_0 - \log r}{n} & \text{if } \mathrm{e}^{-n}\,r_0 < r \leq r_0\,, \\ 0 & \text{if } r > r_0\,. \end{cases} \tag{2.65}$$

Since $|\nabla r| = 1$, we have $|\nabla \eta| \leq \frac{1}{n\,r}$ and $|\nabla \eta|$ is supported in the annulus between $\mathrm{e}^{-n}\,r_0$ and r_0. Applying the stability inequality (i.e., Lemma 1.32)

with the cutoff function η and using the above area bound, we get

$$\begin{aligned}\int_{B^{\Sigma}_{e^{-n}\,r_0}\cap\Sigma}|A|^2 &\leq \int_{\Sigma}\eta^2\,|A|^2 \leq \int_{\Sigma}|\nabla\eta|^2 \leq n^{-2}\int_{B^{\Sigma}_{r_0}\setminus B^{\Sigma}_{e^{-n}\,r_0}} r^{-2} \\ &\leq n^{-2}\sum_{\ell=-n}^{-1}\int_{(B^{\Sigma}_{e^{\ell+1}\,r_0}\setminus B^{\Sigma}_{e^{\ell}\,r_0})} r^{-2} \\ &\leq n^{-2}\sum_{\ell=-n}^{-1}\frac{4}{3}\,\pi\,e^2 = \frac{4\,\pi\,e^2}{3\,n}\,.\end{aligned} \tag{2.66}$$

Therefore, when n is sufficiently large, the intrinsic version of the Choi-Schoen small total curvature estimate (Theorem 2.2 and Remark 2.5) gives (2.64). □

Since intrinsic balls are contained in extrinsic balls:

Corollary 2.11 (R. Schoen [**Sc1**]). *If $\Sigma^2 \subset B_{r_0} = B_{r_0}(x) \subset M^3$ is an immersed stable minimal surface with trivial normal bundle, where $|K_M| \leq k^2$, $r_0 < \rho_1(\pi/k, k)$, and $\partial\Sigma \subset \partial B_{r_0}$, then for some $C = C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.67}$$

R. Schoen has conjectured that this should be true also for stable hypersurfaces in $\mathbb{R}^4$:

Conjecture 2.12 (Schoen). *If $\Sigma^3 \subset \mathbb{R}^4$ is a complete immersed stable minimal hypersurface with trivial normal bundle, then Σ is flat.*

Conjecture 2.13 (Schoen). *If $\Sigma^3 \subset B_{r_0} = B_{r_0}(x) \subset M^4$ is an immersed stable minimal hypersurface with trivial normal bundle where $|K_M| \leq k^2$, $r_0 < \rho_1(\pi/k, k)$, and $\partial\Sigma \subset \partial B_{r_0}$, then for some $C = C(k)$ and all $0 < \sigma \leq r_0$,*

$$\sup_{B_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.68}$$

See [**CM2**] for a generalization of Theorem 2.10 to surfaces that are stable for a parametric elliptic integrand. Before going on, we will briefly discuss this generalization.

Let M^3 be a Riemannian three-manifold. Given a function $\phi \geq 1$ on the unit sphere bundle of M, we can define a functional Φ on an immersed oriented surface Σ in M by

$$\Phi(\Sigma) = \int_{x\in\Sigma}\phi(x, N(x))\,dx\,. \tag{2.69}$$

The restriction to $\phi \geq 1$ is a convenient normalization. By analogy with the area functional (i.e., where $\phi \equiv 1$), a surface is said to be *Φ-stationary* if it is a critical point for Φ. A *Φ-stationary* surface is *Φ-stable* if its second variation is nonnegative for deformations of compact support. See [**Al2**] for more on the first and second variation of a parametric integrand. Φ is *elliptic* if there is some $\lambda > 0$ such that, for each $x \in M$, $v \to [\phi(x, v/|v|) - \lambda]|v|$ is a convex function of $v \in T_x M$.

Curvature estimates continue to hold for Φ-stable surfaces. Namely, let $D_1\phi$ and $D_2\phi$ denote the derivatives of ϕ with respect to the first and second components, respectively; then we have the following:

Theorem 2.14 (Colding-Minicozzi [**CM2**]). *There exist $\epsilon, \rho > 0$, and C, where ρ, C depend on $|\phi|_{C^{2,\alpha}}$, $|D_2\phi|_{C^{2,\alpha}}$, so that if $r_0 \leq \rho$, $|D_2\phi| + |D_2^2\phi| < \epsilon$, and $B_{r_0}^\Sigma = B_{r_0}^\Sigma(x)$ is a ball in an immersed oriented Φ-stable surface $\Sigma \subset \mathbb{R}^3$ with $B_{r_0}^\Sigma \cap \partial\Sigma = \emptyset$, then for all $0 < \sigma \leq r_0$,*

$$\sup_{B^\Sigma_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,. \tag{2.70}$$

As for the minimal surface equation, standard elliptic theory then implies higher derivative estimates (see [**Si2**]; cf. chapter 16 of [**GiTr**]).

We note that many of the standard tools for minimal surfaces do not hold for general Φ. In particular, monotonicity of area (i.e., Proposition 1.12) and Simons' inequality (i.e., Lemma 2.1) no longer hold. This introduces significant complications. For instance, Simons' inequality and an iteration argument were used in [**Sc1**] to estimate the conformal factor for a stable immersion.

3.2. A small excess curvature estimate. We next prove curvature estimates for (smooth) minimal surfaces with small *excess* following [**CM2**]. Allard's regularity theorem [**Al1**] shows that this estimate holds more generally without assuming smoothness.

Let $\Sigma^2 \subset B_1$ be a smooth minimal surface with $\partial\Sigma \subset \partial B_1$. Given $0 < s < 1$ and $x \in B_{1-s} \cap \Sigma$, the *excess* for Σ in $B_s(x)$ is defined to be

$$\Theta_x(s) - 1 = \frac{\text{Area}(B_s(x) \cap \Sigma)}{\pi\, s^2} - 1\,. \tag{2.71}$$

Monotonicity (i.e., Proposition 1.12) implies that the excess is nonnegative and monotone increasing in s.

Theorem 2.15. *There exist $\epsilon > 0$ and $C < \infty$ such that if $0 \in \Sigma^2 \subset B_{2r_0}$ is a smooth compact minimal surface with $\partial\Sigma \subset \partial B_{2r_0}$ and for all $y \in B_{r_0} \cap \Sigma$,*

$$\Theta_y(r_0) - 1 < \epsilon\,, \tag{2.72}$$

then for all $0 < \sigma \le r_0$ *and* $y \in B_{r_0-\sigma}(x)$,

$$\sigma^2 \, |A|^2(y) \le 1 \, . \tag{2.73}$$

Proof. First, note that, by monotonicity, (2.72) implies that for all $s < r_0$ and $y \in B_{r_0} \cap \Sigma$,

$$\Theta_y(s) - 1 < \epsilon \, . \tag{2.74}$$

Defining $F = (r_0 - r)^2 \, |A|^2$ on $B_{r_0} \cap \Sigma$, it suffices to show that $F \le 1$. Suppose not and argue as in the proof of Theorem 2.2, so that after rescaling Σ we have

$$|A|^2(x_0) = \frac{1}{4} \tag{2.75}$$

and

$$\sup_{B_1(x_0)\cap\Sigma} |A|^2 \le 1 \, . \tag{2.76}$$

Furthermore, (2.74) implies that for any $s < 1$,

$$\text{Area}(B_s(x_0) \cap \Sigma) \le \pi(1+\epsilon)s^2 \, . \tag{2.77}$$

To complete the proof, we will show that (2.75), (2.76), and (2.77) lead to a contradiction for $\epsilon > 0$ sufficiently small.

First, applying Theorem 2.2, (2.75) implies that

$$\pi \, C \le \int_{B_{\frac{1}{32}}(x_0)\cap\Sigma} |A|^2 \, , \tag{2.78}$$

where C comes from Theorem 2.2. By Lemma 2.4, $B_{1/8}(x_0) \cap \Sigma$ is a graph with gradient bounded by 1. In particular, it is simply connected and for $x, y \in B_{1/8}(x_0) \cap \Sigma$,

$$|x - y| \le \text{dist}_\Sigma(x, y) \le \sqrt{2} \, |x - y| \, . \tag{2.79}$$

In the remainder of this proof, B^Σ_s will denote the intrinsic ball in Σ of radius s centered at x_0. Combining (2.78) and (2.79), we get

$$\pi \, C \le \int_{B^\Sigma_{1/16}} |A|^2 \, . \tag{2.80}$$

Applying Corollary 2.7 with $r_0 = 1/8$ and $t = 1/32$ gives

$$(1/32)^2 \int_{B^\Sigma_{1/16}} |A|^2 \le 2 \left(\text{Area}(B^\Sigma_{1/8}) - \pi(1/8)^2 \right) , \tag{2.81}$$

so we get that

$$\text{Area}(B^\Sigma_{\frac{1}{8}}) \ge \pi \left(1 + \frac{C}{32}\right) \frac{1}{64} \, . \tag{2.82}$$

Finally, (2.79) and (2.77) give

$$\operatorname{Area}(B^{\Sigma}_{\frac{1}{8}}) \leq \operatorname{Area}(B_{\frac{1}{8}}(x_0) \cap \Sigma) \leq \frac{\pi(1+\epsilon)}{64}, \tag{2.83}$$

which contradicts (2.82) for ϵ sufficiently small. This contradiction completes the proof. □

The above proof shows that, for surfaces, Theorem 2.2 implies the smooth version of Allard's estimate. In fact, these estimates can be shown to be equivalent by a similar argument. This, and related results, are discussed in more detail in [**CM2**].

3.3. Curvature estimates for simply connected surfaces. In fact, for simply connected embedded minimal surfaces, any bound at all on the area or total curvature implies a curvature estimate. This applies more generally to surfaces with quasi-conformal Gauss maps (cf. (7.6)), but we will restrict to minimal surfaces.

The first result of this type was an estimate of Schoen and Simon for extrinsic balls in an embedded minimal disk:

Theorem 2.16 (Schoen-Simon [**ScSi**]). *Let $0 \in \Sigma^2 \subset B_{r_0} = B_{r_0}(0) \subset \mathbb{R}^3$ be an embedded simply connected minimal surface with $\partial \Sigma \subset \partial B_{r_0}$. If $\mu > 0$ and either*

$$\operatorname{Area}(\Sigma) \leq \mu\, r_0^2\,, \quad \textit{or} \tag{2.84}$$

$$\int_{\Sigma} |A|^2 \leq \mu\,, \tag{2.85}$$

then for the connected component Σ' of $B_{r_0/2} \cap \Sigma$ with $0 \in \Sigma'$ we have

$$\sup_{\Sigma'} |A|^2 \leq C\, r_0^{-2} \tag{2.86}$$

for some $C = C(\mu)$.

As an immediate consequence, letting $r_0 \to \infty$ gives Bernstein-type theorems for embedded simply connected minimal surfaces with either bounded density or finite total curvature. Note that Enneper's surface shows that embeddedness is essential.

Instead of proving Theorem 2.16, we will prove a stronger result for intrinsic balls in an embedded minimal disk that is due to Colding and Minicozzi in [**CM4**]. This stronger estimate for intrinsic balls is crucial for applications and immediately gives Theorem 2.16 as a corollary. We will state it only for bounds on $|A|$, but note that it also holds with scale-invariant area bounds because of Corollary 2.7.

Theorem 2.17 (Colding-Minicozzi [**CM4**]). *Given a constant C_I, there exists C_P so that if ${B^\Sigma}_{2s} \subset \Sigma \subset \mathbb{R}^3$ is an embedded minimal disk with*

$$\int_{B_{2s}^\Sigma} |A|^2 \leq C_I \,, \tag{2.87}$$

then

$$\sup_{B_s^\Sigma} |A|^2 \leq C_P \, s^{-2} \,.$$

This theorem will follow easily from the next lemma that gives a curvature estimate for intrinsic balls in an embedded minimal disk with small total curvature in an annulus.

Lemma 2.18 (Colding-Minicozzi [**CM4**]). *Given C, there exists $\epsilon > 0$ so that if ${B^\Sigma}_{9s} \subset \Sigma \subset \mathbb{R}^3$ is an embedded minimal disk with*

$$\int_{{B^\Sigma}_{9s}} |A|^2 \leq C \text{ and } \int_{{B^\Sigma}_{9s} \setminus {B^\Sigma}_s} |A|^2 \leq \epsilon \,, \tag{2.88}$$

then

$$\sup_{{B^\Sigma}_s} |A|^2 \leq s^{-2} \,.$$

Proof. Observe first that for ϵ small, the Choi-Schoen curvature estimate, Theorem 2.2, and (2.88) give

$$\sup_{{B^\Sigma}_{8s} \setminus {B^\Sigma}_{2s}} |A|^2 \leq C_1^2 \, \epsilon \, s^{-2} \,. \tag{2.89}$$

Combining (2.55) and the first inequality in (2.88) gives

$$\text{Length}(\partial {B^\Sigma}_{2s}) \leq (4\pi + C)\, s \,. \tag{2.90}$$

We will next use (2.89) and (2.90) to show that, after rotating $\mathbb{R}^3$, ${B^\Sigma}_{8s} \setminus {B^\Sigma}_{2s}$ is (locally) a graph over $\{x_3 = 0\}$ and, furthermore,

$$\left|\Pi(\partial {B^\Sigma}_{8s})\right| > 3s \,. \tag{2.91}$$

Combining these two facts with embeddedness, the lemma will then follow easily from Rado's theorem.

By (2.90), we have the diameter bound

$$\text{diam}({B^\Sigma}_{8s} \setminus {B^\Sigma}_{2s}) \leq (12 + 2\pi + C/2) s \,.$$

Consequently, since

$$|\nabla \, \text{dist}_{\mathbf{S}^2}(N(x), N)| \leq |A| \,,$$

integrating (2.89) gives

$$\sup_{x, x' \in {B^\Sigma}_{8s} \setminus {B^\Sigma}_{2s}} \text{dist}_{\mathbf{S}^2}(N(x'), N(x)) \leq C_1 \, \epsilon^{1/2} \, (12 + 2\pi + C/2) \,. \tag{2.92}$$

We can therefore rotate $\mathbb{R}^3$ so that

$$\sup_{B^{\Sigma}{}_{8s}\setminus B^{\Sigma}{}_{2s}} |\nabla x_3| \leq C_2\, \epsilon^{1/2}\,(1+C)\,. \tag{2.93}$$

Given $y \in \partial B^{\Sigma}{}_{2s}$, let γ_y be the outward normal geodesic from y to $\partial B^{\Sigma}{}_{8s}$ parametrized by arclength on $[0, 6s]$. Integrating (2.89) along γ_y gives

$$\int_{\gamma_y|_{[0,t]}} |k_g^{\mathbb{R}^3}| \leq \int_{\gamma_y|_{[0,t]}} |A| \leq C_1\, \epsilon^{1/2}\, t/s\,, \tag{2.94}$$

where $k_g^{\mathbb{R}^3}$ is the geodesic curvature of γ_y as a curve in $\mathbb{R}^3$. Since

$$\left| \frac{d}{dt} \langle \gamma_y'(t), \gamma_y'(0)\rangle \right| \leq |k_g^{\mathbb{R}^3}|\,,$$

integrating (2.94) gives a bound for the oscillation of the unit normal γ_y'

$$\langle \gamma_y'(t), \gamma_y'(0)\rangle > 1 - C_3\, \epsilon^{1/2}\,.$$

Integrating again gives that the endpoints of γ_y project to distant points on the line in the $\gamma_y'(0)$ direction

$$\langle \gamma_y(6s) - \gamma_y(0), \gamma_y'(0)\rangle > (1 - C_3\, \epsilon^{1/2})\, 6s\,. \tag{2.95}$$

Since $\gamma_y(0) \in B^{\Sigma}{}_{2s} \subset B_{2s}$ and (2.93) implies that $\gamma_y'(0)$ is nearly horizontal, we see that, for ϵ small, (2.95) implies

$$|\Pi(\partial B^{\Sigma}{}_{8s})| > 3s\,.$$

For the next step, see Figure 2.2. Combining $|\Pi(\partial B^{\Sigma}{}_{8s})| > 3s$ and (2.93), it follows that, for ϵ small,

$$\Pi^{-1}(\partial D_{2s}) \cap B^{\Sigma}{}_{8s}$$

is a collection of immersed multi-valued graphs over ∂D_{2s}. Since $B^{\Sigma}{}_{8s}$ is embedded, $\Pi^{-1}(\partial D_{2s}) \cap B^{\Sigma}{}_{8s}$ consists of disjoint embedded circles which are graphs over ∂D_{2s}; this is the only use of embeddedness. Since $x_1^2 + x_2^2$ is subharmonic on the disk $B^{\Sigma}{}_{8\,s}$, these circles bound disks in $B^{\Sigma}{}_{8\,s}$ which are then graphs by Rado's theorem, i.e., Theorem 1.29.

The curvature bound and gradient bound for the graphs over ∂D_{2s} easily imply that the total geodesic curvature is close to 2π and, thus, Gauss-Bonnet gives that the total curvature on the spanning disks is small (and goes to zero as ϵ does). The lemma follows from Theorem 2.2. □

Proof of Theorem 2.17. Let $\epsilon > 0$ be given by Lemma 2.18 with $C = C_I$ and then let N be the least integer greater than C_I/ϵ. Given $x \in B^{\Sigma}{}_{s}$, there

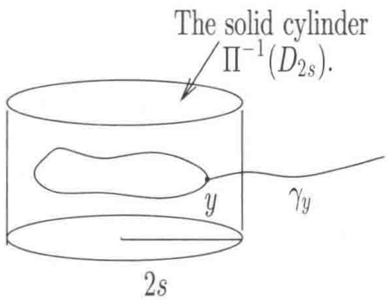

Figure 2.1. Proof of Lemma 2.18: By (2.93) and (2.94), each γ_y is almost a horizontal line segment of length $6s$. Therefore, $|\Pi(\partial B^{\Sigma}{}_{8s})| > 3s$.

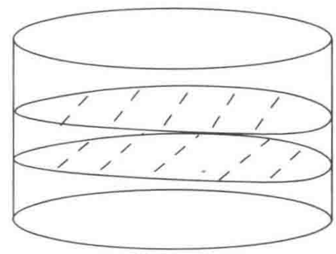

Figure 2.2. Proof of Lemma 2.18: $\Pi^{-1}(\partial D_{2s}) \cap B^{\Sigma}{}_{8s}$ is a union of graphs over ∂D_{2s}. Each bounds a graph in Σ over D_{2s} by Rado's theorem.

exists $1 \leq j \leq N$ with

$$\int_{B^{\Sigma}{}_{9^{1-j}s}(x) \setminus B^{\Sigma}{}_{9^{-j}s}(x)} |A|^2 \leq \frac{C_I}{N} \leq \epsilon \,. \tag{2.96}$$

Combining (2.87) and (2.96), Lemma 2.18 gives that

$$|A|^2(x) \leq (9^{-j}s)^{-2} \leq 9^{2N}\, s^{-2} \,. \qquad \square$$

It seems likely that there is a generalization of this to higher dimensions, though clearly one will need to assume more than simply connected. We conjecture that:

Conjecture 2.19. *A complete embedded minimal hypersurface in $\mathbb{R}^4$ that is contractible and has cubic volume growth must be flat.*

This conjecture would of course follow from the following stronger conjecture:

Conjecture 2.20. *Prove a curvature estimate for intrinsic balls in an embedded minimal hypersurface in $\mathbb{R}^4$ that are contractible and have bounded volume. (Namely, generalize Theorem* 2.17.*)*

4. L^p Bounds of $|A|^2$ for Stable Hypersurfaces

In this section, we combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. In the next section, we will use this bound together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces.

Theorem 2.21 (Schoen-Simon-Yau [**ScSiYa**]). *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is an orientable stable minimal hypersurface. For all $p \in [2, 2+\sqrt{2/(n-1)})$ and each nonnegative Lipschitz function ϕ with compact support*

$$\int_\Sigma |A|^{2p}\,\phi^{2p} \leq C(n,p) \int_\Sigma |\nabla \phi|^{2p}\,. \tag{2.97}$$

Proof. If we insert $\eta = |A|^{1+q} f$ in the stability inequality (i.e., Lemma 1.32) for $0 \leq q < \sqrt{2/(n-1)}$, we get

$$\begin{aligned} \int |A|^{4+2q}\,f^2 &\leq \int |f\,\nabla |A|^{1+q} + |A|^{1+q}\,\nabla f|^2 \\ &= (1+q)^2 \int f^2 |\nabla |A||^2\,|A|^{2q} + \int |A|^{2+2q}|\nabla f|^2 \\ &\quad + 2(1+q)\int f|A|^{1+2q}\langle \nabla f, \nabla |A|\rangle\,. \end{aligned} \tag{2.98}$$

We will use this inequality twice. The key step is the first use of (2.98) where we combine it with Simons' inequality to bound the term on the right involving $|\nabla|A||^2$. We then substitute this bound into (2.98) to prove the theorem.

Multiply Simons' inequality

$$|A|\,\Delta|A| + |A|^4 \geq \frac{2}{n-1}|\nabla |A||^2\,, \tag{2.99}$$

by $|A|^{2q} f^2$ and integrate to get

$$\begin{aligned} \frac{2}{n-1}\int |\nabla|A||^2\,|A|^{2q} f^2 &\leq \int |A|^{4+2q} f^2 + \int f^2 |A|^{1+2q}\Delta|A| \\ &= \int |A|^{4+2q} f^2 - 2\int f|A|^{1+2q}\langle \nabla f, \nabla |A|\rangle \\ &\quad - (1+2q)\int f^2 |A|^{2q} |\nabla |A||^2\,. \end{aligned} \tag{2.100}$$

Combining (2.98) and (2.100) gives

$$\left(\frac{2}{n-1} - q^2\right)\int |A|^{2q}\,|\nabla|A||^2\,f^2 \tag{2.101}$$

$$\leq \int |A|^{2+2q}\,|\nabla f|^2 + 2q\int f|A|^{1+2q}\,\langle \nabla f,\, \nabla|A|\rangle \tag{2.102}$$

$$\int |A|^{2+2q}\,|\nabla f|^2 + \epsilon\, q \int f^2\,|A|^{2q}\,|\nabla|A||^2 + \frac{q}{\epsilon}\int |\nabla f|^2\,|A|^{2+2q}\,,$$

where the last inequality used the absorbing inequality $2xy \leq \epsilon\, x^2 + y^2/\epsilon$. Hence, we get that

$$\left(\frac{2}{n-1} - q^2 - \epsilon\, q\right)\int f^2|A|^{2q}|\nabla|A||^2 \leq \left(1 + \frac{q}{\epsilon}\right)\int |\nabla f|^2 |A|^{2+2q}\,. \tag{2.103}$$

This completes the first step.

Applying the Cauchy-Schwarz inequality to absorb the cross-term in (2.98) and then substituting (2.103) gives (for any $\epsilon < \frac{2/(n-1)-q^2}{q}$)

$$\int |A|^{4+2q} f^2 \leq 2(1+q)^2 \int f^2 |A|^{2q} |\nabla |A||^2 + 2 \int |A|^{2q+2} |\nabla f|^2$$

$$\leq \left(\frac{2(1+q)^2(1+q/\epsilon)}{\frac{2}{n-1} - q^2 - \epsilon\, q} + 2 \right) \int |A|^{2+2q} |\nabla f|^2 . \tag{2.104}$$

If we set $p = 2 + q$ and $f = \phi^p$, then $2 \leq p < 2 + \sqrt{2/(n-1)}$ and Hölder's inequality gives

$$\int |A|^{2p} \phi^{2p} \leq c \int |A|^{2p-2} \phi^{2p-2} |\nabla \phi|^2$$

$$\leq c \left(\int |A|^{2p} \phi^{2p} \right)^{\frac{p-1}{p}} \left(\int |\nabla \phi|^{2p} \right)^{\frac{1}{p}} , \tag{2.105}$$

for some $c = c(n,p)$. □

5. Bernstein Theorems and Curvature Estimates

In this section, we will first show a generalization of the Bernstein theorem proven in Chapter 1.

Theorem 2.22 (Schoen-Simon-Yau, [**ScSiYa**]). *If $\Sigma^{n-1} \subset \mathbb{R}^n$ is a complete two-sided stable minimal hypersurface, $n \leq 6$, and there exists $V < \infty$ so that*

$$\sup_{R>0} \frac{\mathrm{Vol}(B_R \cap \Sigma)}{R^{n-1}} \leq V , \tag{2.106}$$

then Σ is flat.

Proof. For each $r > 0$ let ϕ be the cutoff function with $\phi | B_r \equiv 1$, $\phi | B_{2r} \equiv 0$ and such that ϕ decays linearly in the radial direction on the annulus $B_{2r} \setminus B_r$. By combining the L^p bound of $|A|^2$ from Theorem 2.21 for this cutoff function and

$$2p = 4 + \sqrt{7/5} < 4 + \sqrt{8/(n-1)} \tag{2.107}$$

with the volume bound (2.106), we get

$$\int_{B_r \cap \Sigma} |A|^{4+\sqrt{7/5}} \leq C(n,p)\, r^{-4-\sqrt{7/5}}\, \mathrm{Vol}(B_{2r} \cap \Sigma)$$

$$\leq C(n,p)\, 2^{n-1}\, V\, r^{n-5-\sqrt{7/5}} . \tag{2.108}$$

Since $n - 5 - \sqrt{7/5} < 0$, by letting r go to infinity we get that $|A|^2 \equiv 0$. The claim now easily follows. □

Since minimal graphs are stable and, by (1.22), satisfy the area bound (2.106), we get the following Bernstein theorem:

Theorem 2.23 (S. Bernstein [**Be**] for $n = 3$, E. De Giorgi [**DG**] for $n = 4$, F. J. Almgren, Jr. [**Am1**] for $n = 5$, and J. Simons [**Sim**] for $n \leq 8$). *If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is an entire solution to the minimal surface equation and $n \leq 8$, then u is a linear function.*

Proof. When $n \leq 6$, this follows immediately from Theorem 2.22. See [**Sim**] for $n = 7$ and 8. □

In fact, a variation on this argument gives curvature estimates for stable minimal surfaces with bounded area and for minimal graphs. We will give the statement only for graphs, but the statement and proofs are similar in the stable case except that one must assume the area bound. Namely, we have the following:

Theorem 2.24 (E. Heinz [**He**] for $n = 3$, Schoen-Simon-Yau [**ScSiYa**] for $n \leq 6$, and L. Simon [**Si1**] for $n \leq 8$). *If $u : D_{r_0} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is a solution to the minimal surface equation on the $(n-1)$-dimensional disk of radius r_0 and $n \leq 8$, then for $0 < \sigma \leq r_0$,*

$$\sup_{D_{r_0 - \sigma}} |A|^2 \leq C(n)\, \sigma^{-2} . \tag{2.109}$$

Proof. (Again, we prove this only for $n \leq 6$.) The proof closely follows that of Theorem 2.3. Namely, taking $2p = n$ in Theorem 2.21 and letting ϕ be a logarithmic cutoff function (as in the proof of Corollary 1.20), we get that the area bounds for minimal graphs imply that

$$\int_{B_{\sqrt{\kappa}\, R} \cap \mathrm{Graph}_u} |A|^n \leq \frac{C_1}{\log \kappa} , \tag{2.110}$$

where $C_1 = C_1(n)$, $\kappa > 1$, and $\kappa\, R \leq r_0$. On the other hand, Simons' inequality implies that

$$\Delta |A|^n \geq -C_2\, |A|^{n+2} , \tag{2.111}$$

so we can argue as in Theorem 2.2 to get a corresponding curvature estimate. □

6. The General Minimal Graph Equation

The main result of this section, Proposition 2.25, gives a useful criterion for stability of a minimal surface $\Sigma \subset \mathbb{R}^3$. The rough idea is that a minimal surface that can be approached on one side by a sequence of minimal surfaces must be stable. A quantitative version of this result played an important role in analyzing the local structure of embedded minimal surfaces in [**CM4**].

To make this precise, define the graph of a function u over a surface Σ by

$$\{x + u(x)\, N(x) \,|\, x \in \Sigma\}\,, \tag{2.112}$$

where N is a unit normal to Σ. When Σ is a plane, this is just the usual notion of a graph.

Proposition 2.25. *Let Σ be a minimal surface. If there is a sequence u_j so the graphs of u_j over Σ are minimal, the u_j's are never zero, and $|u_j| + |\nabla u_j| \to 0$, then Σ is stable.*

The key point for proving the proposition is that the functions u_j nearly satisfy the Jacobi equation on Σ.

Lemma 2.26. *There is a constant C so that if Σ is a minimal surface, the graph of u over Σ is also minimal, and $\max\{|u|\,|A|\,, |\nabla u|\} \leq 1$, then u satisfies the equation*

$$\operatorname{div}\left[(I + \bar{L})\nabla u\right] + (1 + Q)\,|A|^2\, u + Q_{ij} A_{ij} = 0\,, \tag{2.113}$$

where $\bar{L} \leq C\,(|u|\,|A| + |\nabla u|)$, $|Q| \leq C\,(|u|\,|A| + |\nabla u|)^2$, and $|Q_{ij}| \leq C\,(|u|\,|A| + |\nabla u|)^2$.

Proof. Let x be the position vector, N the unit normal for Σ, and let $\{e_1, e_2\}$ be an orthonormal frame along Σ. We will extend u, e_1, e_2, and N to a small neighborhood of Σ by making all of them constant in the normal direction. Thus, ∇u is tangent to Σ and

$$\nabla_N\, e_i = \nabla_N N = 0\,. \tag{2.114}$$

Set $A_{ij} = \langle A(e_i, e_j), N\rangle$, so that at any point in Σ we have

$$\nabla_{e_i} N = -A_{ij}\, e_j\,. \tag{2.115}$$

Using the minimality of Σ and the symmetry of A, it is easy to see that

$$A_{ij} A_{jk} = \frac{1}{2}\,|A|^2\,\delta_{ik}\,. \tag{2.116}$$

The graph of u over Σ is given by

$$x \to X(x) = x + u(x)\, N\,, \tag{2.117}$$

so the tangent is spanned by X_1 and X_2 where

$$X_i = \nabla_{e_i} X = e_i + u_i\, N - u A_{ik}\, e_k\,. \tag{2.118}$$

Here we used (2.115) to differentiate N and we used that $\nabla_v x = v$ for any vector v.

Using (2.118), we compute the metric g_{ij} for the graph:

$$
\begin{aligned}
g_{ij} &= \langle X_i, X_j \rangle = \delta_{ij} + u_i u_j + u^2 A_{ik} A_{jk} - 2\, u A_{ij} \\
&= \left(1 + \frac{|A|^2}{2}\, u^2 \right) \delta_{ij} + u_i u_j - 2\, u A_{ij} \,. \qquad (2.119)
\end{aligned}
$$

Since we will consider solutions where $|\nabla u| + |u|\,|A|$ is small, we will often write this as

$$g_{ij} = \delta_{ij} - 2\, u A_{ij} + Q_{ij} \,, \qquad (2.120)$$

where Q_{ij} denotes a matrix that is quadratic in $|\nabla u|$ and $|u|\,|A|$ (and will be allowed to vary from line to line). More precisely, there exists C so that if $\max\{|u|\,|A|\,, |\nabla u|\} \leq 1$, then

$$|Q_{ij}(x)| \leq C\,(|u|\,|A| + |\nabla u|)^2 \,. \qquad (2.121)$$

It follows that the inverse metric is given by

$$g^{ij} = \delta_{ij} + 2\, u A_{ij} + Q_{ij} \,. \qquad (2.122)$$

It follows from (2.120) that

$$\det g_{ij} = (1 - 2uA_{11})(1 - 2uA_{22}) + Q = 1 + Q \,, \qquad (2.123)$$

where the second equality used minimality (i.e., $A_{11} + A_{22} = 0$) and Q this time is a scalar quadratic term. Whenever $J(s)$ is a differentiable path of matrices, the derivative at 0 of the determinant is given by

$$\frac{d}{ds}\Big|_{s=0} \det J(s) = \det J(0)\, \mathrm{Trace}\left(J^{-1}(0)\, J'(0) \right) \,. \qquad (2.124)$$

Applying this with $J(s) = g_{ij}(s)$ where $g_{ij}(s)$ is computed with $(u + sv)$ in place of u, we get

$$
\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \det g_{ij} \\
&\quad = (1 + Q)\, (\delta_{ij} + 2\, u A_{ij} + Q_{ij})\, \left(|A|^2 uv \delta_{ij} - 2v\, A_{ij} + u_i v_j + u_j v_i \right) \\
&\quad = 2\langle \nabla u, \nabla v \rangle - 2\,(1 + Q)\,|A|^2\, uv + Q_{ij} A_{ij}\, v + \langle \bar{L} \nabla u, \nabla v \rangle \,, \qquad (2.125)
\end{aligned}
$$

where in the last term $\bar{L}$ is a matrix that is linearly bounded in $|\nabla u| + |A|\,|u|$ (again assuming that $\max\{|u|\,|A|\,, |\nabla u|\} \leq 1$) and we also used $(1+Q)\, Q_{ij} = Q_{ij}$. Hence, if the graph of u is minimal, then we get for every v with compact support that

$$0 = \int_\Sigma \langle (I + \bar{L}) \nabla u, \nabla v \rangle - (1 + Q)\, |A|^2\, uv + Q_{ij} A_{ij} v \,. \qquad (2.126)$$

Integrating by parts to take ∇ off of v, we get that u satisfies (2.113). □

Proof of Proposition 2.25. By Proposition 1.39, it suffices to construct a nonvanishing Jacobi field w on an arbitrary open connected subset $K_0 \subset \Sigma$ with compact closure. Fix a point p in the interior of K_0 and let K_1 be a connected open set containing K_0 that also has compact closure.

Since $|A|$ is bounded on compact subsets of Σ and

$$|u_j| + |\nabla u_j| \to 0\,, \tag{2.127}$$

Lemma 2.26 gives C so that u_j satisfies

$$\operatorname{div}\left[(I + \bar{L}_j)\nabla u_j\right] + (1 + Q_j)\,|A|^2\, u_j + \bar{Q}_j = 0\,, \tag{2.128}$$

where

$$\left|\bar{L}_j\right|^2 + |Q_j| + |\bar{Q}_j| \le C\,(|u|\,|A| + |\nabla u|)^2\,. \tag{2.129}$$

In particular, the u_j's satisfy a uniform Harnack inequality:

$$\sup_{K_1} u_j \le C_1 \inf_{K_1} u_j \le C_1\, u_j(p)\,. \tag{2.130}$$

In two dimensions, the above equation also implies Holder continuity of the gradient of u_j (see theorem 12.4 in [**GiTr**]) which then allows us to apply the Schauder estimates (theorem 6.2 in [**GiTr**]) to get

$$||u_j||_{C^{2,\alpha}(K_0)} \le C_2\, u_j(p)\,. \tag{2.131}$$

In particular, if we define functions w_j by

$$w_j(x) = \frac{u_j(x)}{u_j(p)}\,, \tag{2.132}$$

then each $w_j(p) = 1$, the w_j's are positive, and

$$||w_j||_{C^{2,\alpha}(K_0)} \le C_2\,. \tag{2.133}$$

We can therefore apply Azela-Ascoli to extract a subsequence (still called w_j) that converges uniformly in C^2 to a limiting function w with $w(p) = 1$ and $w \ge 0$. Multiplying (2.128) by $1/u_j(p)$ and using the uniform convergence, we can pass to limits to get that

$$\operatorname{div} \nabla w + |A|^2\, w = 0\,, \tag{2.134}$$

where we used (2.129) and the fact that $u_j + |\nabla u_j| \to 0$ to get rid of the higher order terms in the limit. Finally, w also has a Harnack inequality so it must also be positive. $\square$

There are quantitative versions of this too, where one shows that if two minimal surfaces with bounded curvature come close, but don't touch, then each of them is almost stable. We explore this next.

7. Almost Stability

The estimates for stable minimal surfaces can be extended to surfaces that are "almost stable" in a precise sense of [**CM4**] that we give below. This extension plays a crucial role since it is possible to show that nearby, but disjoint, minimal surfaces with bounded curvature are almost stable, but are not necessarily stable in the usual sense. The results of this section are taken from [**CM4**].

We begin with a generalization of stability:

Definition 2.27 (δ_s-stability). *Given $\delta_s \geq 0$, set*

$$L_{\delta_s} = \Delta + (1-\delta_s)|A|^2 \,, \tag{2.135}$$

so L_0 is the usual Jacobi operator on Σ. A domain $\Omega \subset \Sigma$ is δ_s-stable if

$$\int \phi \, L_{\delta_s} \phi \leq 0$$

for any compactly supported Lipschitz function ϕ (i.e., $\phi \in C^{0,1}_0(\Omega)$).

It follows that Ω is δ_s-stable if and only if, for all $\phi \in C^{0,1}_0(\Omega)$, we have the δ_s-stability inequality:

$$(1-\delta_s) \int |A|^2 \phi^2 \leq \int |\nabla \phi|^2 \,. \tag{2.136}$$

Since the Jacobi equation is the linearization of the minimal graph equation over Σ by Lemma 2.26, standard calculations give:

Lemma 2.28. *There exists $\delta_g > 0$ so that if Σ is minimal and u is a positive solution of the minimal graph equation over Σ (i.e., $\{x + u(x)\, N_\Sigma(x) \,|\, x \in \Sigma\}$ is minimal) with*

$$|\nabla u| + |u|\,|A| \leq \delta_g \,,$$

then $w = \log u$ satisfies on Σ

$$\Delta w = -|\nabla w|^2 + \operatorname{div}(a \nabla w) + \langle \nabla w, a \nabla w \rangle + \langle b, \nabla w \rangle + (c-1)|A|^2 \,, \tag{2.137}$$

for functions a_{ij}, b_j, c on Σ with $|a|, |c| \leq 3\,|A|\,|u| + |\nabla u|$ and $|b| \leq 2\,|A|\,|\nabla u|$.

Proof. This follows easily from Lemma 2.26. □

The following slight modification of Proposition 1.39 gives a useful sufficient condition for δ_s-stability of a domain:

Lemma 2.29 (Colding-Minicozzi, [**CM4**]). *There exists $\delta > 0$ so that if Σ is minimal and $u > 0$ is a solution of the minimal graph equation over $\Omega \subset \Sigma$ with*

$$|\nabla u| + |u|\,|A| \leq \delta \,,$$

then Ω is $1/2$-stable.

Proof. Set $w = \log u$ and choose a cutoff function $\phi \in C_0^{0,1}(\Omega)$. Applying Stokes' theorem to

$$\operatorname{div}(\phi^2 \nabla w - \phi^2 a \nabla w),$$

substituting (2.137), and using $|a|, |c| \leq 3\delta, |b| \leq 2\delta |\nabla w|$ gives

$$\begin{aligned}(1 - 3\delta) \int \phi^2 |A|^2 &\leq - \int \phi^2 |\nabla w|^2 + \int \phi^2 \langle \nabla w, b + a \nabla w \rangle \\ &\quad + 2 \int \phi \langle \nabla \phi, \nabla w - a \nabla w \rangle \\ &\leq (5\delta - 1) \int \phi^2 |\nabla w|^2 + 2(1 + 3\delta) \int |\phi \nabla w| |\nabla \phi| . \end{aligned} \tag{2.138}$$

The lemma follows from the absorbing inequality $2xy \leq \epsilon x^2 + y^2/\epsilon$. □

We will use Lemma 2.29 to see that disjoint embedded minimal surfaces that are close are nearly stable (Corollary 2.31 below).

We will need a sharpening of Lemma 2.4 which showed that bounds on $|A|$ give graphical regions. As in the proof of Lemma 2.4, the starting point is integrating

$$|\nabla \operatorname{dist}_{\mathbf{S}^2}(N(x), N)| \leq |A| \tag{2.139}$$

on geodesics to get

$$\sup_{x' \in B^{\Sigma}{}_s(x)} \operatorname{dist}_{\mathbf{S}^2}(N(x'), N(x)) \leq s \sup_{B^{\Sigma}{}_s(x)} |A| . \tag{2.140}$$

By (2.140), we can choose $0 < \rho_2 < 1/4$ so:

> If $B^{\Sigma}{}_{2s}(x) \subset \Sigma$, $s \sup_{B^{\Sigma}{}_{2s}(x)} |A| \leq 4\rho_2$, and $t \leq s$, then the component $\Sigma_{x,t}$ of $B_t(x) \cap \Sigma$ with $x \in \Sigma_{x,t}$ is a graph over $T_x\Sigma$ with gradient $\leq t/s$ and

$$\inf_{x' \in B^{\Sigma}{}_{2s}(x)} \frac{|x' - x|}{\operatorname{dist}_{\Sigma}(x, x')} > 9/10 . \tag{2.141}$$

One consequence is that if $t \leq s$ and we translate $T_x\Sigma$ so that $x \in T_x\Sigma$, then

$$\sup_{x' \in B^{\Sigma}{}_t(x)} |x' - T_x\Sigma| \leq t^2/s . \tag{2.142}$$

The next lemma shows that two disjoint minimal surfaces with bounded curvature that come near each other can be written as graphs over each other of functions that nearly satisfy the Jacobi equation. The lemma is stated so that it can be applied either to two everywhere disjoint surfaces, or to two subsets of a single embedded surface where the subdomains are intrinsically far enough apart.

Lemma 2.30 (Colding-Minicozzi, [**CM4**]). *There exist $C_0, \rho_0 > 0$ so that the following holds:*

If $\rho_1 \leq \min\{\rho_0, \rho_2\}$ and $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ are oriented minimal surfaces with $|A|^2 \leq 4$ on each Σ_i so that

$$x \in \Sigma_1 \setminus \mathcal{T}_{4\rho_2}(\partial \Sigma_1)\,,$$

$$y \in B_{\rho_1}(x) \cap \Sigma_2 \setminus \mathcal{T}_{4\rho_2}(\partial \Sigma_2)\,,$$

$$B^{\Sigma}{}_{2\rho_1}(x) \cap B^{\Sigma}{}_{2\rho_1}(y) = \emptyset\,,$$

then $B^{\Sigma}{}_{\rho_2}(y)$ is the graph $\{z + u(z)\, N(z)\}$ over a domain containing $B^{\Sigma}{}_{\rho_2/2}(x)$ with $u \neq 0$ and $|\nabla u| + 4\,|u| \leq C_0\, \rho_1$.

Proof. Since $\rho_1 \leq \rho_2$, (2.141) implies that

$$B^{\Sigma}{}_{2\rho_2}(x) \cap B^{\Sigma}{}_{2\rho_2}(y) = \emptyset\,.$$

If $t \leq 9\rho_2/5$, then $|A|^2 \leq 4$ implies that the components $\Sigma_{x,t}, \Sigma_{y,t}$ of $B_t(x) \cap \Sigma_1, B_t(y) \cap \Sigma_2$, respectively, with $x \in \Sigma_{x,t}, y \in \Sigma_{y,t}$, are graphs with gradient $\leq t/(2\rho_2)$ over $T_x\Sigma_1, T_y\Sigma_2$ and have

$$\Sigma_{x,t} \subset B^{\Sigma}{}_{2\rho_2}(x),\ \Sigma_{y,t} \subset B^{\Sigma}{}_{2\rho_2}(y)\,.$$

The last conclusion implies that $\Sigma_{x,t} \cap \Sigma_{y,t} = \emptyset$. It now follows that $\Sigma_{x,t}, \Sigma_{y,t}$ are graphs over the same plane. Namely, if we set

$$\theta = \operatorname{dist}_{\mathbf{S}^2}(N(x), \{N(y), -N(y)\})\,,$$

then (2.142), $|x - y| < \rho_1$, and $\Sigma_{x,t} \cap \Sigma_{y,t} = \emptyset$ imply that

$$\rho_1 - (t/2 - \rho_1)\sin\theta + t^2/(2\rho_2) > -t^2/(2\rho_2)\,. \tag{2.143}$$

Hence,

$$\sin\theta < \rho_1/(t/2 - \rho_1) + t^2/[(t/2 - \rho_1)\rho_2]\,.$$

For ρ_0/ρ_2 small, $B^{\Sigma}{}_{\rho_2}(y)$ is a graph with bounded gradient over $T_x\Sigma_1$. The lemma now follows easily using the Harnack inequality. □

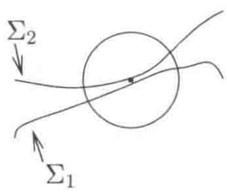

Figure 2.3. Corollary 2.31: Two sufficiently close disjoint minimal surfaces with bounded curvatures must be nearly stable.

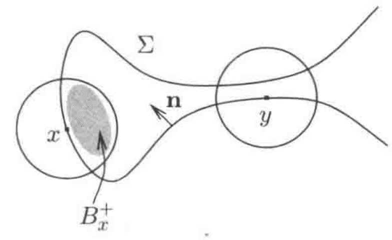

Figure 2.4. The set VB in (2.148). Here $x \in VB$ and $y \in \Sigma \setminus VB$.

Combining Lemmas 2.29 and 2.30 gives the next corollary:

Corollary 2.31 (Colding-Minicozzi, [**CM4**]). *See Figure 2.3. Given C_0, $\delta > 0$, there exists $\epsilon(C_0, \delta) > 0$ so that if $p_i \in \Sigma_i \subset \mathbb{R}^3$ $(i = 1, 2)$ are embedded minimal surfaces with*

$$\Sigma_1 \cap \Sigma_2 = \emptyset\,,\ B^{\Sigma}{}_{2R}(p_i) \cap \partial \Sigma_i = \emptyset\,,\ |p_1 - p_2| < \epsilon\, R$$

and

$$\sup_{B^{\Sigma}{}_{2R}(p_i)} |A|^2 \leq C_0\, R^{-2}\,, \tag{2.144}$$

then $B^{\Sigma}{}_R(\tilde{p}_i) \subset \tilde{\Sigma}_i$ is δ-stable where $\tilde{p}_i$ is the point over p_i in the universal cover $\tilde{\Sigma}_i$ of Σ_i.

7.1. Finding large almost stable regions. The next result gives a decomposition of an embedded minimal surface with bounded curvature into a portion with bounded area and a union of disjoint 1/2-stable domains. This result is more technical, but will be useful in later chapters when we study the local structure of embedded minimal disks near a point of large curvature (as in [**CM4**]).

Lemma 2.32 (Colding-Minicozzi, [**CM4**]). *There exists C_1 so that:*
If $0 \in \Sigma \subset B_{2R} \subset \mathbb{R}^3$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{2R}$, and $|A|^2 \leq 4$, then there exist disjoint 1/2-stable subdomains $\Omega_j \subset \Sigma$ and a function $\chi \leq 1$ which vanishes on $B_R \cap \Sigma \setminus \cup_j \Omega_j$ so that

$$\text{Area}(\{x \in B_R \cap \Sigma \,|\, \chi(x) < 1\}) \leq C_1\, R^3\,, \tag{2.145}$$

$$\int_{B^{\Sigma}{}_R} |\nabla \chi|^2 \leq C_1\, R^3\,. \tag{2.146}$$

Proof. We can assume that $R > \rho_2$ (otherwise $B_R \cap \Sigma$ is stable). Let $\delta > 0$ be from Lemma 2.29 and C_0, ρ_0 be from Lemma 2.30. Set

$$\rho_1 = \min\left\{\frac{\rho_0}{C_0}, \frac{\delta}{C_0}, \frac{\rho_2}{4}\right\}. \tag{2.147}$$

Given $x \in B_{2R-\rho_1} \cap \Sigma$, let Σ_x be the component of $B_{\rho_1}(x) \cap \Sigma$ with $x \in \Sigma_x$ and let B_x^+ be the component of $B_{\rho_1}(x) \setminus \Sigma_x$ which $N(x)$ points into. See Figure 2.4. Set

$$VB = \{x \in B_R \cap \Sigma \,|\, B_x^+ \cap \Sigma \setminus {B^\Sigma}_{4\rho_1}(x) = \emptyset\} \tag{2.148}$$

and let $\{\Omega_j\}$ be the components of $B_R \cap \Sigma \setminus \overline{VB}$. Choose a maximal disjoint collection

$$\{{B^\Sigma}_{\rho_1}(y_i)\}_{1 \leq i \leq \nu} \tag{2.149}$$

of balls centered in VB. Hence, the union of the balls

$$\{{B^\Sigma}_{2\rho_1}(y_i)\}_{1 \leq i \leq \nu} \tag{2.150}$$

covers the set VB. Further, the "half-balls"

$$B_{\rho_1/2}(y_i) \cap B_{y_i}^+ \tag{2.151}$$

are pairwise disjoint. To see this, suppose that $|y_i - y_j| < \rho_1$ but $y_j \notin {B^\Sigma}_{2\rho_1}(y_i)$. Then, by (2.141), $y_j \notin {B^\Sigma}_{8\rho_1}(y_i)$ so ${B^\Sigma}_{4\rho_1}(y_j) \notin B_{y_i}^+$ and ${B^\Sigma}_{4\rho_1}(y_i) \notin B_{y_j}^+$; the triangle inequality then implies that

$$B_{\rho_1/2}(y_i) \cap B_{y_i}^+ \cap B_{\rho_1/2}(y_j) \cap B_{y_j}^+ = \emptyset\,,$$

as claimed. By (2.140)–(2.142), we see that each $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ has volume approximately ρ_1^3 and is contained in B_{2R} so that $\nu \leq C\,R^3$. Define the function χ on Σ by

$$\chi(x) = \begin{cases} 0 & \text{if } x \in VB\,, \\ \operatorname{dist}_\Sigma(x, VB)/\rho_1 & \text{if } x \in \mathcal{T}_{\rho_1}(VB) \setminus VB\,, \\ 1 & \text{otherwise}\ . \end{cases} \tag{2.152}$$

Since $\mathcal{T}_{\rho_1}(VB) \subset \bigcup_{i=1}^{\nu} {B^\Sigma}_{3\rho_1}(y_i)$, $|A|^2 \leq 4$, and $\nu \leq C\,R^3$, we get (2.145). Combining (2.145) and $|\nabla \chi| \leq \rho_1^{-1}$ gives (2.146) (taking C_1 larger).

It remains to show that each Ω_j is $1/2$-stable. Fix j. By construction, if $x \in \Omega_j$, then there exists $y_x \in B_x^+ \cap \Sigma \setminus {B^\Sigma}_{4\rho_1}(x)$ minimizing $|x - y_x|$ in $B_x^+ \cap \Sigma$. In particular, by Lemma 2.30, ${B^\Sigma}_{\rho_2}(y_x)$ is the graph $\{z + u_x(z)\,N(z)\}$ over a domain containing ${B^\Sigma}_{\rho_2/2}(x)$ with $u_x > 0$ and

$$|\nabla u_x| + 4\,|u_x| \leq \min\{\delta, \rho_0\}\,. \tag{2.153}$$

Choose a maximal disjoint collection of balls ${B^{\Sigma}}_{\rho_2/6}(x_i)$ with $x_i \in \Omega_j$ and let $u_{x_i} > 0$ be the corresponding functions defined on ${B^{\Sigma}}_{\rho_2/2}(x_i)$. Since Σ is embedded (and compact) and $|u_{x_i}| < \rho_0$, Lemma 2.30 implies that

$$u_{x_i}(x) = \min_{t>0}\{x + t\, N(x) \in \Sigma\}$$

for $x \in {B^{\Sigma}}_{\rho_2/2}(x_i)$. Hence, $u_{x_{i_1}}(x) = u_{x_{i_2}}(x)$ for $x \in {B^{\Sigma}}_{\rho_2/2}(x_{i_1}) \cap {B^{\Sigma}}_{\rho_2/2}(x_{i_2})$. Note that $\mathcal{T}_{\rho_2/6}(\Omega_j) \subset \bigcup_i {B^{\Sigma}}_{\rho_2/2}(x_i)$. We conclude that the u_{x_i}'s give a well defined function $u_j > 0$ on $\mathcal{T}_{\rho_2/6}(\Omega_j)$ with

$$|\nabla u_j| + |u_j|\,|A| \leq \delta\,. \tag{2.154}$$

Finally, Lemma 2.29 implies that each Ω_j is 1/2-stable. □

8. Sublinear Growth of the Separation

As we have seen, the separation is constant for the multi-valued graphs coming from each half of the helicoid. This can be viewed as a type of Liouville Theorem reflecting the conformal properties of an infinite-valued graph. In [**CM3**], we gave a corresponding gradient estimate for positive solutions of the minimal graph equation defined on a multi-valued graph:

Theorem 2.33 (Colding-Minicozzi, [**CM3**]). *Given $\alpha > 0$, there exist $\delta_p > 0, N_g > 5$ so that the following holds:*

If Σ is a N_g-valued minimal graph of a function u on $D_{e^{N_g} R} \setminus D_{e^{-N_g} R}$ with $|\nabla u| \leq 1$ and $0 < w < \delta_p R$ is a solution of the minimal graph equation over Σ with $|\nabla w| \leq 1$, then for $R \leq s \leq 2R$,

$$\sup_{\Sigma^{0,2\pi}_{R,2R}} |A_\Sigma| + \sup_{\Sigma^{0,2\pi}_{R,2R}} |\nabla w|/w \leq \alpha/(4\,R)\,, \tag{2.155}$$

$$\sup_{\Sigma^{0,2\pi}_{R,s}} w \leq (s/R)^\alpha \sup_{\Sigma^{0,2\pi}_{R,R}} w\,. \tag{2.156}$$

We will prove this theorem in this section.

One important consequence of (2.156) is that, for $\alpha < 1$, the separation grows sublinearly. This plays an important role in [**CM3**] for extending multi-valued graphs and again in [**CM6**] in the proof of the one-sided curvature estimate (see Chapter 8). See [**CM8**] for sharper logarithmic estimates for multi-valued graphs with a growing number of sheets; cf. [**CM29**]. When the multi-valued graph is part of an embedded minimal disk, then one can say much more; see [**CM8**].

Let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$.

8.1. An analog for harmonic functions. Before we prove Theorem 2.33, we will give an indication of why it is true in this subsection.

The function w is almost a solution of the linearized equation (which is the Jacobi equation in this case). Since the graphs have bounded gradient, this equation is not too far from the Laplace equation. To give an indication of why (2.156) holds, we will give an elementary proof when u and w are harmonic.

Proof of Theorem 2.33 when u is harmonic. After rescaling, we can assume that $R = 1$. By making the conformal change of coordinates

$$(\rho, \theta) \to (\log \rho, \theta) \tag{2.157}$$

we get a positive harmonic function

$$\tilde{w}(x, y) = w(\mathrm{e}^x, y) \tag{2.158}$$

defined on the square $[-N_g, N_g] \times [-N_g, N_g]$. Since the chain rule gives

$$\nabla \log w(1,0) = \nabla \log \tilde{w}(0,0)\,, \tag{2.159}$$

applying the Euclidean gradient estimate to $\tilde{w}$ yields

$$|\nabla \log w(1,0)| = |\nabla \log \tilde{w}(0,0)| \leq C/N_g\,. \tag{2.160}$$

This gives the sublinear gradient estimate for w if N_g is sufficiently large. The bound on Hess_u follows similarly. □

8.2. Constructing good cutoff functions. The next lemma and corollary construct the cutoff function needed in the gradient estimate Theorem 2.33. The material from this subsection and the next is all from [**CM3**].

Lemma 2.34. *Given $N > 36/(1-\mathrm{e}^{-1/3})^2$, there exists a function $0 \leq \phi \leq 1$ on $\mathcal{P}$ with $\int |\nabla \phi|^2 \leq 4\pi/\log N$ and*

$$\phi = \begin{cases} 1 & \text{if } R/\mathrm{e} \leq \rho \leq \mathrm{e}\, R \text{ and } |\theta| \leq 3\pi\,, \\ 0 & \text{if } \rho \leq \mathrm{e}^{-N} R \text{ or } \mathrm{e}^N R \leq \rho \text{ or } |\theta| \geq \pi N\,. \end{cases} \tag{2.161}$$

Proof. After rescaling, we may assume that $R = 1$. Since energy is conformally invariant on surfaces, composing with z^{3N} implies that (2.161) is equivalent to $E(\phi) \leq 4\pi/\log N$ and

$$\phi = \begin{cases} 1 & \text{if } |\log \rho| < 1/(3N) \text{ and } |\theta| \leq \pi/N\,, \\ 0 & \text{if } |\log \rho| > 1/3 \text{ or } |\theta| \geq \pi/3\,. \end{cases} \tag{2.162}$$

This is achieved (with $E(\phi) = 2\pi/\log[N(1-\mathrm{e}^{-1/3})/6]$) by setting

$$\phi = \begin{cases} 1 & \text{on } D_{6/N}(1,0)\,, \\ 1 - \frac{\log[N\,\mathrm{dist}_{\mathcal{P}}((1,0),\cdot)/6]}{\log[N(1-\mathrm{e}^{-1/3})/6]} & \text{on } D_{1-\mathrm{e}^{-1/3}}(1,0)\setminus D_{6/N}(1,0)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.163}$$

□

Given an N-valued graph Σ, let $\Sigma_{r_3,r_4}^{\theta_1,\theta_2} \subset \Sigma$ be the subgraph over

$$\{(\rho,\theta)\,|\,r_3 \le \rho \le r_4,\ \theta_1 \le \theta \le \theta_2\}\,. \tag{2.164}$$

Transplanting the cutoff function from Lemma 2.34 to a multi-valued graph gives the next corollary:

Corollary 2.35. *Given $\epsilon_0, \tau > 0$, there exists $N > 0$ so if $\Sigma \subset \mathbb{R}^3$ is an N-valued graph over $D_{\mathrm{e}^N R} \setminus D_{\mathrm{e}^{-N} R}$ of u with $|\nabla u| \le \tau$, then there is a cutoff function $0 \le \phi \le 1$ on Σ with $\int |\nabla\phi|^2 \le \epsilon_0$, $\phi|_{\partial\Sigma} = 0$, and*

$$\phi \equiv 1 \text{ on } \Sigma_{R/2,5R/2}^{-\pi,3\pi}\,. \tag{2.165}$$

Proof. Since $\Sigma_{R/2,5R/2}^{-\pi,3\pi} \subset \Sigma_{R/\mathrm{e},\mathrm{e}R}^{-3\pi,3\pi}$ and the projection from Σ to $\mathcal{P}$ is bi-Lipschitz with bi-Lipschitz constant bounded by $\sqrt{1+\tau^2}$, the corollary follows from Lemma 2.34. □

8.3. The gradient estimate. If $u > 0$ is a solution of the Jacobi equation $\Delta u = -|A|^2 u$ on Σ, then $w = \log u$ satisfies

$$\Delta w = -|\nabla w|^2 - |A|^2\,. \tag{2.166}$$

The Bochner formula (Proposition 1.47), (2.166), $\mathrm{K}_\Sigma = -|A|^2/2$, and the Cauchy-Schwarz inequality give

$$\begin{aligned} \Delta|\nabla w|^2 &= 2\,|\,\mathrm{Hess}_w\,|^2 + 2\langle \nabla w, \nabla \Delta w\rangle - |A|^2\,|\nabla w|^2 \\ &\ge 2\,|\,\mathrm{Hess}_w\,|^2 - 4\,|\nabla w|^2\,|\,\mathrm{Hess}_w\,| - 4\,|\nabla w|\,|A|\,|\nabla A| - |A|^2\,|\nabla w|^2 \\ &\ge -2\,|\nabla w|^4 - 3\,|A|^2\,|\nabla w|^2 - 2\,|\nabla A|^2\,. \end{aligned} \tag{2.167}$$

Using that the Jacobi equation is the linearization of the minimal graph equation over Σ, Lemma 2.28 gave that analogs of (2.166) and (2.167) hold for solutions of the minimal graph equation over Σ.

Proof of Theorem 2.33. Fix $\epsilon_E > 0$ (to be chosen depending only on α). Corollary 2.35 gives N (depending only on ϵ_E) and a function $0 \le \phi \le 1$ with compact support on $\Sigma_{\mathrm{e}^{-N} R,\mathrm{e}^N R}^{-N\pi,N\pi}$,

$$\int |\nabla\phi|^2 \le \epsilon_E \text{ and } \phi \equiv 1 \text{ on } \Sigma_{R/2,5R/2}^{-\pi,3\pi}\,. \tag{2.168}$$

Set $N_g = N + 1$, so that

$$\operatorname{dist}_\Sigma(\Sigma^{-N\pi,N\pi}_{\mathrm{e}^{-N}R,\mathrm{e}^{N}R}, \partial\Sigma) > \mathrm{e}^{-N}\, R/2$$

and hence Heinz's curvature estimate gives that

$$|A_\Sigma| \leq C\,\mathrm{e}^{N}/R$$

on $\Sigma^{-N\pi,N\pi}_{\mathrm{e}^{-N}R,\mathrm{e}^{N}R}$.

Now fix $x \in \Sigma^{0,2\pi}_{R,2R}$. Since multi-valued graphs are stable by Lemma 1.36, we can use ϕ in the stability inequality to get

$$\int_{\Sigma^{-\pi,3\pi}_{R/2,5R/2}} |A|^2 \leq \int_\Sigma |A_\Sigma|^2\,\phi^2 \leq \int |\nabla\phi|^2 \leq \epsilon_E\,,$$

where the last inequality used (2.168). Hence, we can apply the mean value inequality to get

$$\sup_{B^\Sigma_{3R/8}(x)} (R^2\,|\nabla A_\Sigma|^2 + |A_\Sigma|^2) \leq C\,\epsilon_E R^{-2}\,. \tag{2.169}$$

Since Σ and the graph of w are (locally) graphs with bounded gradient, it is easy to see that

$$\sup_{\Sigma^{-N\pi,N\pi}_{\mathrm{e}^{-N}R,\mathrm{e}^{N}R}} |\nabla w| \leq C\,\mathrm{e}^{N} \sup_\Sigma |w|/R \leq C\,\mathrm{e}^{N}\,\delta_p\,. \tag{2.170}$$

Set $v = \log w$. Choose $\delta_p > 0$ (depending only on N), so that (2.170) implies that w satisfies (2.137) on $\Sigma^{-N\pi,N\pi}_{\mathrm{e}^{-N}R,\mathrm{e}^{N}R}$ with

$$|a|, |b|/|A|, |c| \leq 1/4\,.$$

Applying Stokes' theorem to

$$\operatorname{div}(\phi^2\nabla v - \phi^2 a\nabla v)$$

and using the absorbing inequality gives

$$\int_{B^\Sigma_{R/2}(x)} |\nabla v|^2 \leq \int \phi^2|\nabla v|^2 \leq C\,E(\phi) \leq C\,\epsilon_E\,. \tag{2.171}$$

Combining (2.137) and (2.169), an easy calculation (as in (2.167)) shows that on $B^\Sigma_{3R/8}(x)$

$$\Delta|\nabla v|^2 \geq -C\,|\nabla v|^4 - C\,\epsilon_E\,R^{-2}\,|\nabla v|^2 - C\,\epsilon_E\,R^{-4}\,. \tag{2.172}$$

By the rescaling argument used to prove Theorem 2.2, (2.171) and (2.172) imply a pointwise bound for $|\nabla v|^2$ on $B^\Sigma_{R/4}(x)$; combining this with (2.169) gives (2.155) for ϵ_E small.

To get the (2.156), integrate $|\nabla \log |w|| \leq \alpha/\rho$ along each ray $\theta = \theta_0$ to get

$$\log \frac{w(r_2, \theta_0)}{w(r_1, \theta_0)} \leq \alpha \int_{r_1}^{r_2} \rho^{-1}\, d\rho = \log \left(\frac{r_2}{r_1}\right)^{\alpha}, \tag{2.173}$$

and then use the elementary inequality

$$(s - R)/R \leq 2 \log(s/R). \qquad \square$$

9. Minimal Cones

We close this chapter with a discussion of minimal cones in Euclidean space. The study of these cones has been important both in the generalizations of the theorem of Bernstein and on issues of local regularity.

If N^{k-1} is a (smooth) submanifold of $\mathbf{S}^{n-1} \subset \mathbb{R}^n$, then the *cone* over N will be denoted by $C(N)$ and, as a set,

$$C(N) = \{x \in \mathbb{R}^n \mid x/|x| \in N\}. \tag{2.174}$$

Therefore $C(N)$ is a smooth k-dimensional submanifold away from the origin (which is the vertex of the cone). By definition, cones are invariant under dilations about the origin. First, we have the following simple lemma whose proof is left for the reader:

Lemma 2.36. *A submanifold $N^{k-1} \subset \mathbf{S}^{n-1}$ is minimal if and only if its Euclidean mean curvature is everywhere normal to $\mathbf{S}^{n-1} \subset \mathbb{R}^n$.*

Let $x = (x_1, \ldots, x_n)$ denote the position vector in $\mathbb{R}^n$ and set $r = |x|$. It follows from this lemma that if $N^{k-1} \subset \mathbf{S}^{n-1}$ is minimal, then the Euclidean mean curvature of N is given by the vector field

$$\Delta x = (\Delta x_1, \ldots, \Delta x_n), \tag{2.175}$$

where Δ is the metric Laplacian on N.

We now get the following lemma:

Lemma 2.37. *If $N^{k-1} \subset \mathbf{S}^{n-1}$ is a minimal submanifold, then the coordinate functions are eigenfunctions with eigenvalue $k - 1$.*

Proof. Since N is minimal, Δx is normal to $\mathbf{S}^{n-1}$ and, thus, points in the same direction as the position vector x. It follows that

$$\Delta x = x\, f \tag{2.176}$$

for some function f. Since $x \in \mathbf{S}^{n-1}$, $|x|^2 = 1$ and hence

$$0 = \Delta |x|^2 = 2\, \langle x, \Delta x \rangle + 2 |\nabla x|^2 = 2\, f + 2(k-1), \tag{2.177}$$

so the function f is equal to the constant $(1 - k)$, giving the lemma. $\square$

Example 2.38. If $\mathbf{S}^{k-1}$ is a totally geodesic $(k-1)$-sphere in $\mathbf{S}^{n-1}$, then $C(\mathbf{S}^{k-1})$ is a k-dimensional plane through the origin in $\mathbb{R}^n$.

Let $N^{k-1} \subset \mathbf{S}^{n-1}$ be an immersed submanifold (not necessarily minimal). Since

$$\nabla_{C(N)} u = r^{-1} \nabla_N u(r^{-1} \cdot) + \frac{\partial u}{\partial r} \frac{\partial}{\partial r}, \tag{2.178}$$

a direct computation shows that the Laplacians of N and $C(N)$ are related by the following simple formula at $x \neq 0$:

$$\Delta_{C(N)} u = r^{-2} \Delta_N u(r^{-1} x) + (k-1)\, r^{-1} \frac{\partial}{\partial r} u + \frac{\partial^2}{\partial r^2} u. \tag{2.179}$$

Lemma 2.39. *If $N^{k-1} \subset \mathbf{S}^{n-1}$ is a minimal submanifold, then $C(N) \subset \mathbb{R}^n$ is minimal.*

Proof. It suffices to show that each coordinate function x_i is harmonic on $C(N)$. We write $x_i = r\, u_i$ where u_i is independent of r and so that x_i and u_i agree on $N \subset \mathbf{S}^{n-1}$. Lemma 2.37 and (2.179) give

$$\begin{aligned} \Delta_{C(N)} x_i &= r^{-1} \Delta_N u_i + u_i\, (k-1)\, r^{-1} \frac{\partial}{\partial r} r + u_i \frac{\partial^2}{\partial r^2} r \\ &= -(k-1)\, r^{-1} u_i + (k-1)\, r^{-1} u_i = 0\,. \end{aligned} \tag{2.180}$$ □

Minimal cones arise in the generalizations of Bernstein's original theorem and in the study of local regularity for minimal submanifolds. The key fact here is that since cones are invariant under dilations, the ratio

$$\frac{\mathrm{Vol}(C(N) \cap B_r)}{r^k} \tag{2.181}$$

is constant.

Conversely, if Σ^k is minimal and this ratio is constant, then we have equality in the monotonicity formula. It follows that $\nabla |x|^2$ is tangent to Σ almost everywhere and hence that Σ is invariant under dilations.

Suppose now that Σ^k is an area-minimizing minimal hypersurface. Arguing as we did in (1.22) for minimal graphs in Chapter 1, the density V_Σ is uniformly bounded. If $r_j \to 0$, then the sequence $\Sigma_j = r_j\, \Sigma$ of dilated surfaces is also area-minimizing. The bound on V_{Σ_j} implies the existence of a convergent subsequence (say, as area-minimizing currents or stationary varifolds); see the next chapter for more on this. The (not necessarily unique) limit is denoted by Σ_∞ and is also area-minimizing. It follows from the monotonicity formula that $\mathrm{V}_{\Sigma_\infty} = \mathrm{V}_\Sigma$ is also equal to the density of Σ_∞ at the origin. Therefore, we have equality in the monotonicity formula and consequently Σ_∞ is an area-minimizing minimal cone. This argument,

which dates back to W. Fleming [**Fl**], allows one to prove the Bernstein theorem by showing the nonexistence of area-minimizing cones.

Theorem 2.40 (Fleming, [**Fl**]). *If there is a nonlinear entire solution of the minimal surface equation on $\mathbb{R}^n$, then there is a singular area-minimizing cone in $\mathbb{R}^{n+1}$.*

Shortly after Fleming's result, E. De Giorgi showed in [**DG**] that the minimal cone given by Theorem 2.40 was in fact cylindrical. Here cylindrical means that the cone, as a subset of $\mathbb{R}^n$, splits off a line isometrically, that is,

$$C(N) = \mathbb{R} \times C(N') \subset \mathbb{R}^n . \tag{2.182}$$

This implied that there actually existed a singular area-minimizing cone in $\mathbb{R}^n$.

The significance of this approach is that cones are much simpler to analyze. This connection is fundamental in the modern theory of regularity; an important illustration of this is the so-called dimension reduction argument of Federer. For instance, in dimension two such a cone must be a collection of planes through the origin; it is easy to see that this can only be minimizing when there is just a single plane. F. J. Almgren, Jr. showed that in dimension three such a cone was also a hyperplane. Finally, J. Simons proved the same theorem for $n \leq 7$.

Theorem 2.41 (F. J. Almgren, Jr. [**Am1**] for $n = 3$ and J. Simons [**Sim**] for $n \leq 7$). *The hyperplanes are the only stable minimal hypercones in $\mathbb{R}^n$ for $n \leq 7$.*

However, in 1969 Bombieri, De Giorgi, and Giusti [**BDGG**] gave an example of an area-minimizing singular cone in $\mathbb{R}^8$. In fact, they showed that for $m \geq 4$ the cones

$$C_m = \{(x_1, \dots, x_{2m}) \mid x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\} \subset \mathbb{R}^{2m} \tag{2.183}$$

are area-minimizing.

Weak Convergence, Compactness and Applications

In this chapter, we will introduce a notion of weak convergence, called varifold convergence, for submanifolds where the submanifold is identified with a measure that one can integrate against. We will see that there is a natural generalization of minimal submanifolds and that much of the classical theory can be extended to this setting. We will next prove the Michael-Simon Sobolev inequality. We will then prove a generalization from [**CM16**] of the classical Bernstein theorem for minimal surfaces discussed in the previous chapters.

The various Bernstein theorems imply that, through dimension seven, area-minimizing hypersurfaces must be affine. A weaker form of this is true in all dimensions by the Allard regularity theorem [**All**]. Namely, there exists

$$\delta = \delta(k, n) > 0 \tag{3.1}$$

such that if $\Sigma^k \subset \mathbb{R}^n$ is a k-dimensional complete immersed minimal submanifold with

$$\mathrm{Vol}(B_r(x) \cap \Sigma) \leq (1 + \delta)\ \mathrm{Vol}(B_r \subset \mathbb{R}^k) \tag{3.2}$$

for all x and r, then Σ is an affine k-plane. More generally, this theorem holds even when Σ has singularities, for instance, when Σ is a stationary integral k-varifold (see Definitions 3.3, 3.4, and 3.10 below).

In this chapter, we will show that, in fact, a bound on the density (see (3.34) for the definition of the density for a varifold) gives an upper bound for the dimension of the smallest affine subspace containing the minimal surface. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally, from properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space. We refer to the original paper [**CM16**] for further discussion and more general results.

Finally, in the last section of this chapter, we will discuss another notion of weak convergence, called bubble convergence, that is related to work of Sacks and Uhlenbeck and Gromov. We will then show that bubble convergence implies varifold convergence.

1. The Theory of Varifolds

We will begin by briefly describing the basic theory of varifolds. For convenience of notation, we will restrict ourselves to varifolds in $\mathbb{R}^n$; without much further work we could consider varifolds in smooth n-dimensional manifolds. Varifolds should be thought of as a generalization of submanifolds. This class of generalized submanifolds is broad enough to give general compactness results and allow for singularities but is narrow enough to still have geometric significance. For instance, we can define the first variation for a varifold.

In the theory of varifolds, surfaces are identified with certain Radon measures. The basic compactness result will then follow from the compactness theorem for Radon measures.

Definition 3.1. Let X be a locally compact separable space. A *Radon measure* μ is a Borel regular measure which is finite on compact subsets of X.

The following standard compactness result will be the main benefit of viewing submanifolds as Radon measures:

Theorem 3.2 (Compactness of Radon Measures). *Let X be a locally compact separable space and μ_j a sequence of Radon measures on X such that*

$$\sup_j \mu_j(U) < \infty \tag{3.3}$$

for any open set U with compact closure. Then there exists a Radon measure μ and a subsequence $\mu_{j'}$ such that $\mu_{j'} \to \mu$; that is, if f is a continuous function on X with compact support, then

$$\lim_{j' \to \infty} \int_X f \, d\mu_{j'} = \int_X f \, d\mu \, . \tag{3.4}$$

Let $\Sigma^k \subset \mathbb{R}^n$ be a smooth properly embedded submanifold. There is an obvious way to associate to Σ a Radon measure μ; namely, let

$$\mu(A) = \mathrm{Vol}(A \cap \Sigma) \tag{3.5}$$

on a Borel set $A \subset \mathbb{R}^n$. Theorem 3.2 gives compactness for submanifolds (or the associated measures) as long as the submanifolds have uniform local area bounds. That is, suppose that $C < \infty$ and $\Sigma_j \subset \mathbb{R}^n$ is a sequence of k-dimensional submanifolds with

$$\mathrm{Vol}(B_1(x) \cap \Sigma_j) < C \tag{3.6}$$

for all $x \in \mathbb{R}^n$ and all j. We define associated Radon measures μ_j as in (3.5). By Theorem 3.2, there exists a subsequence j' and a Radon measure μ so that the $\mu_{j'}$ converge weakly to μ.

The price that we have to pay for viewing a sequence of submanifolds as measures is that the limit is merely a measure (and need not exhibit any geometry). In particular, if we were attempting to extract an area-minimizing submanifold from a minimizing sequence, then we would not be able to say that the limit was minimal (i.e., had zero first variation).

Let $G(k,n)$ be the space of (unoriented) k-planes through the origin in $\mathbb{R}^n$ (so that $G(n-1,n)$ is projective $(n-1)$ space). Let

$$\pi : \mathbb{R}^n \times G(k,n) \to \mathbb{R}^n \tag{3.7}$$

denote projection $\pi(x,\omega) = x$. Note that π is proper. We will also define the divergence and gradient with respect to a k-dimensional plane ω at $x \in \mathbb{R}^n$ as follows: If X is a C^1 vector field, then we define the divergence with respect to (x,ω) by

$$\mathrm{div}_\omega X = \sum_{i=1}^{k} \langle E_i, \nabla_{E_i} X \rangle\,, \tag{3.8}$$

where E_i is an orthonormal basis for ω and ∇ is the Euclidean derivative. Likewise, if u is a C^1 function, then we set

$$\nabla_\omega u = \sum_{i=1}^{k} E_i(u)\, E_i\,. \tag{3.9}$$

Definition 3.3 (Varifold). A *k-varifold* T on $\mathbb{R}^n$ is a Radon measure on $\mathbb{R}^n \times G(k,n)$. The weight μ_T of T is the Radon measure on $\mathbb{R}^n$ given by $\mu_T(E) = T(\pi^{-1}E)$, the support of T is the support of μ_T, and the mass of T on a set $U \subset \mathbb{R}^n$ is just $\mu_T(U)$.

Once again, let $\Sigma^k \subset \mathbb{R}^n$ be a smooth properly embedded submanifold. Define $T\Sigma \subset \mathbb{R}^n \times G(k,n)$ by $(x,\omega) \in T\Sigma$ if ω is the tangent k-plane to Σ

at x. Σ gives rise to a k-varifold T defined by

$$T(A) = \mathrm{Vol}(\pi(A \cap T\Sigma)) \tag{3.10}$$

on a Borel set $A \subset \mathbb{R}^n \times G(k,n)$. Here Vol is k-dimensional volume. In this case, the mass of T on a set U is simply the volume of Σ in U.

With this definition, it is easy to see how varifolds behave under mappings. Namely, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 mapping (so that F and dF are C^1), then F induces a C^1 map $\mathcal{F}$ from $\mathbb{R}^n \times G(k,n)$ to $\mathbb{R}^n \times \bigcup_{\ell \leq k} G(\ell, n)$:

$$\mathcal{F}(x,\omega) = (F(x), dF_x(\omega)) \,. \tag{3.11}$$

If T is a k-varifold and the mapping F is proper on the support of T, then we define the (weighted) push-forward varifold $F(T)$ by

$$F(T)(A) = \int_{\mathcal{F}^{-1}A} J_\omega F(x)\, dT(x,\omega)\,, \tag{3.12}$$

where $J_\omega F(x)$ is the Jacobian factor given by

$$J_\omega F(x) = \left(\det (dF_x|_\omega)^t \circ (dF_x|_\omega)\right)^{\frac{1}{2}} \,. \tag{3.13}$$

The integrand (3.13) vanishes on the set where $\mathcal{F}^{-1}$ is not defined, and hence the push-forward varifold $F(T)(A)$ given by (3.12) is well defined. When T comes from a smooth surface Σ, $J_\omega F(x)$ is just the square root of the determinant of the pullback metric and then (3.12) is just the usual area formula (see, for instance, [**Fe**]).

Using this transformation rule for varifolds, we can compute the first variation of "mass" for a varifold. Suppose that

$$F(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \tag{3.14}$$

is a one-parameter family of C^2 diffeomorphisms such that $F(\cdot,0)$ is the identity and $F(\cdot,t)$ is the identity outside of a fixed compact set K. Let $F_t(x)$ denote the vector field $\frac{\partial F}{\partial t}(x,0)$ (so that F_t vanishes outside a compact set K) and suppose that $K \subset U \subset \mathbb{R}^n$. Differentiating under the integral sign in (3.12) and arguing precisely as in Chapter 1, we get

$$\frac{d}{dt}_{t=0} \mu_{F(T,t)}(U) = \int_{U \times G(k,n)} \mathrm{div}_\omega\, F_t(x)\, dT(x,\omega)\,. \tag{3.15}$$

We will say that a varifold is stationary if it has zero first variation for any compactly supported C^1 vector field.

Definition 3.4. A varifold T is *stationary* if for any compactly supported C^1 vector field X we have

$$\int_{\mathbb{R}^n \times G(k,n)} \mathrm{div}_\omega\, X\, dT(x,\omega) = 0\,. \tag{3.16}$$

Note that, by (3.15), this definition is equivalent to requiring that the varifold is a critical point for the mass functional for all compactly supported C^2 variations. Observe also that if a varifold arises from a smooth submanifold, then the first variation formula implies that it is stationary if and only if it has zero mean curvature.

An important consequence of this definition is the following supplement to the compactness theorem for varifolds discussed earlier:

Proposition 3.5. *If T_j is a sequence of stationary varifolds which converges weakly to a varifold T, then T is also stationary.*

Proof. Given a C^1 vector field X with compact support, we can define a compactly supported continuous function f by

$$f(x,\omega) = \operatorname{div}_\omega X\,. \tag{3.17}$$

Since the T_j are stationary,

$$\int_{\mathbb{R}^n \times G(k,n)} f(x,\omega)\, dT_j(x,\omega) = \int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega X\, dT_j(x,\omega) = 0\,. \tag{3.18}$$

Therefore by varifold convergence and by the compact support of f on $\mathbb{R}^n \times G(k,n)$, we get

$$\begin{aligned}\int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega X\, dT(x,\omega) &= \int_{\mathbb{R}^n \times G(k,n)} f\, dT(x,\omega)\\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n \times G(k,n)} f\, dT_j(x,\omega) = 0\,.\end{aligned} \tag{3.19}$$

□

Many of the earlier results about smooth minimal surfaces can be extended to stationary varifolds by inserting appropriate choices of vector fields in (3.16). We will give the arguments for two of these extensions below: the harmonicity of the coordinate functions and the monotonicity formula. These results should be compared with those of Chapter 1, and with Propositions 1.7, 1.12, and 1.15, in particular.

A function u on $\mathbb{R}^n$ is said to be *weakly harmonic* on a varifold T if, for any smooth function η with compact support,

$$\int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega u\rangle\, dT(x,\omega) = 0\,. \tag{3.20}$$

Similarly we say that a function u on $\mathbb{R}^n$ is *weakly subharmonic* on a varifold T if, for any smooth nonnegative function η with compact support,

$$\int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega u\rangle\, dT(x,\omega) \le 0\,. \tag{3.21}$$

Proposition 3.6 (Harmonicity of the Coordinate Functions). *A varifold $T^k \subset \mathbb{R}^n$ is a stationary varifold if and only if the coordinate functions x_i are weakly harmonic.*

Proof. Let η be a smooth function with compact support and set $e_i = \nabla x_i$. For any unit vector E_j we have $\nabla_{E_j}(\eta\, e_i) = (\nabla_{E_j}\eta) e_i$, and hence for any k-plane ω

$$\operatorname{div}_\omega(\eta\, e_i) = \langle \nabla_\omega \eta, e_i\rangle = \langle \nabla_\omega \eta, \nabla_\omega x_i\rangle\,, \tag{3.22}$$

where $\nabla_\omega \eta$ is the projection of $\nabla \eta$ to ω. Therefore,

$$\int_{\mathbb{R}^n \times G(k,n)} \operatorname{div}_\omega(\eta\, e_i)\, dT(x,\omega) = \int_{\mathbb{R}^n \times G(k,n)} \langle \nabla_\omega \eta, \nabla_\omega x_i\rangle\, dT(x,\omega)\,. \tag{3.23}$$

The claim easily follows from (3.23). □

Similarly, the monotonicity formula may be generalized to this setting. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a nonnegative function and set $r = |x|$. Given any vector E, we have

$$\nabla_E x = E\,, \tag{3.24}$$

where $x = (x_1, \ldots, x_n)$. For ω a k-plane with orthonormal basis E_i, we use (3.24) to compute

$$\begin{aligned}\operatorname{div}_\omega(\eta(r)x) &= \sum_{i=1}^{k} \langle E_i, \nabla_{E_i}(\eta(r)x)\rangle \\ &= k\,\eta(r) + \eta'(r)\,\langle \nabla_\omega r, x\rangle = k\,\eta(r) + r\,\eta'(r)\,|\nabla_\omega r|^2\,.\end{aligned} \tag{3.25}$$

Let ω^N denote the orthogonal $(n-k)$-plane to ω. We have

$$1 = |\nabla r|^2 = |\nabla_\omega r|^2 + |\nabla_{\omega^N} r|^2\,. \tag{3.26}$$

Proposition 3.7 (Monotonicity for Stationary Varifolds). *Suppose that $T^k \subset \mathbb{R}^n$ is a stationary varifold and $x_0 \in \mathbb{R}^n$; then for all $0 < s < t$*

$$\begin{aligned} &t^{-k}\,\mu_T(B_t(x_0)) - s^{-k}\,\mu_T(B_s(x_0)) \\ &\qquad = \int_{(B_t(x_0)\setminus B_s(x_0))\times G(k,n)} r^{-k}\,|\nabla_{\omega^N} r|^2\, dT(x,\omega)\,.\end{aligned} \tag{3.27}$$

Proof. After a translation, it suffices to assume that $x_0 = 0$. Let ϕ be a nonnegative cutoff function with $\phi'(s) \leq 0$ which is identically one on $[0, \frac{1}{2}]$ and supported on $[0,1]$. Fix s for the moment and let $\eta(r) = \phi(\frac{r}{s})$ so that

$$r\,\eta'(r) = -s\,\frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\,. \tag{3.28}$$

Since T is stationary, integrating (3.25) and using (3.26) gives

$$(3.29)\qquad \begin{aligned} 0 &= \int \operatorname{div}_\omega(\eta(r)x)\, dT(x,\omega) \\ &= \int \left(k\,\eta(r) + r\,\eta'(r)\right)\, dT(x,\omega) - \int r\,\eta'(r)\, |\nabla_{\omega^N} r|^2\, dT(x,\omega)\,. \end{aligned}$$

Substituting (3.28) into (3.29) gives

$$(3.30)\qquad \begin{aligned} &\int k\,\phi\left(\frac{r}{s}\right) - s\,\frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\, dT(x,\omega) \\ &\qquad = -s\int \frac{d}{ds}\left(\phi\left(\frac{r}{s}\right)\right)\, |\nabla_{\omega^N} r|^2\, dT(x,\omega)\,. \end{aligned}$$

Multiplying through by s^{-k-1}, we may rewrite (3.30) as

$$(3.31)\qquad \begin{aligned} &\frac{d}{ds}\left(s^{-k}\int \phi\left(\frac{r}{s}\right)\, dT(x,\omega)\right) \\ &\qquad = s^{-k}\frac{d}{ds}\left(\int \phi\left(\frac{r}{s}\right)\, |\nabla_{\omega^N} r|^2\, dT(x,\omega)\right)\,. \end{aligned}$$

If we let ϕ increase to the characteristic function of $[0,1]$ and apply the monotone convergence theorem to (3.31), we get

$$(3.32)\qquad \begin{aligned} \frac{d}{ds}\left(s^{-k}\,\mu_T(B_s)\right) &= s^{-k}\frac{d}{ds}\left(\int_{B_s\times G(k,n)} |\nabla_{\omega^N} r|^2\, dT(x,\omega)\right) \\ &= \frac{d}{ds}\left(\int_{B_s\times G(k,n)} r^{-k}\, |\nabla_{\omega^N} r|^2\, dT(x,\omega)\right)\,. \end{aligned}$$

Equation (3.32) holds both in the sense of distributions and for almost every s. To see this, we use the monotonicity of both $\mu_T(B_s)$ and the integral on the right-hand side of (3.32). Integrating (3.32) from s to t yields

$$(3.33)\qquad \begin{aligned} &t^{-k}\,\mu_T(B_t) - s^{-k}\,\mu_T(B_s) \\ &\qquad = \int_{(B_t\setminus B_s)\times G(k,n)} r^{-k}\, |\nabla_{\omega^N} r|^2\, dT(x,\omega)\,. \qquad \square \end{aligned}$$

In Chapter 1, we defined the density for a minimal submanifold. We can now extend this to stationary varifolds. Namely, given a stationary k-varifold $T^k \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, and $s > 0$, we define the density by

$$(3.34)\qquad \Theta_{x_0}(s) = \frac{\mu_T(B_s(x_0))}{\operatorname{Vol}(B_s \subset \mathbb{R}^k)}\,.$$

A simple modification of the argument in Proposition 1.12 implies that $\Theta_{x_0}(s)$ is a nondecreasing function of s.

Similarly, the more general mean value inequality also extends to stationary varifolds. The necessary modifications will be left to the reader.

Proposition 3.8 (The Mean Value Inequality for Stationary Varifolds). *If $T \subset \mathbb{R}^n$ is a stationary k-varifold, $x_0 \in \mathbb{R}^n$, f is a nonnegative weakly subharmonic function on T, and $0 < s < t$, then*

$$t^{-k} \int_{B_t(x_0)} f \, d\mu_T \geq s^{-k} \int_{B_s(x_0)} f \, d\mu_T \,. \tag{3.35}$$

Thus far we have been discussing the theory of general varifolds. The varifolds which come from smooth submanifolds have a good deal of additional structure. In particular, the associated measures are supported on smooth submanifolds which have tangent spaces at every point. The class of varifolds known as rectifiable varifolds have similar additional structure.

Let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure.

Definition 3.9. A set $S \subset \mathbb{R}^n$ is said to be *k-rectifiable* if $S \subset S_0 \cup S_1$, where $\mathcal{H}^k(S_0) = 0$ and S_1 is the image of $\mathbb{R}^k$ under a Lipschitz map.

More generally, S is said to be *countably k-rectifiable* if $S \subset \bigcup_{\ell \geq 0} S_\ell$, where $\mathcal{H}^k(S_0) = 0$ and for $\ell \geq 1$ each S_ℓ is the image of $\mathbb{R}^k$ under a Lipschitz map.

One advantage of working with rectifiable sets is that they have tangent spaces at almost every point by Rademacher's theorem [**Fe**]. Consequently, we may associate a varifold to a rectifiable set, just as we did for smooth submanifolds. Rectifiable varifolds are then varifolds which are supported on rectifiable sets.

Definition 3.10 (Rectifiable Varifold). Let S be a countably k-rectifiable subset of $\mathbb{R}^n$ with $\mathcal{H}^k(S) < \infty$ and let θ be a positive locally $\mathcal{H}^k$ integrable function on S. Set T equal to the varifold associated to the set S (exactly as if S were a smooth submanifold). The associated varifold $T' = \theta \, T$ is called a *rectifiable varifold.* If θ is integer-valued, then T' is an *integral varifold.*

Henceforth, $\Sigma^k \subset \mathbb{R}^n$ will be a stationary rectifiable k-varifold. Associated to each such Σ is a Radon measure μ_Σ; abusing notation slightly, we will also use Σ to denote the (Hausdorff k-dimensional rectifiable) set on which the measure is supported. We shall make the standard assumption that the density is at least 1 on the support. Note that this class of generalized minimal submanifolds includes the case of embedded minimal submanifolds equipped with the intrinsic Riemannian metric.

1.1. Varifold distance. Fix a closed manifold M and let

$$\Pi : G_k M \to M$$

be the Grassmanian bundle of (un-oriented) k-planes, that is, each fiber $\Pi^{-1}(p)$ is the set of all k-dimensional linear subspaces of the tangent space of M at p. Since G_kM is compact, we can choose a countable dense subset $\{h_n\}$ of all continuous functions on G_kM with supremum norm at most one (dense with respect to the supremum norm).[1] If (X_0, F_0) and (X_1, F_1) are two compact (not necessarily connected) surfaces X_0, X_1 with measurable maps

$$F_i : X_i \to G_kM$$

so that each $f_i = \Pi \circ F_i$ is in the Sobolev space $W^{1,2}(X_i, M)$ (of maps that are square-integrable and whose gradients are also square-integrable) and J_{f_i} is the Jacobian of f_i, then the varifold distance between them is by definition

$$d_V(F_0, F_1) = \sum_n 2^{-n} \left| \int_{X_0} h_n \circ F_0 \, J_{f_0} - \int_{X_1} h_n \circ F_1 \, J_{f_1} \right| . \tag{3.36}$$

It follows easily that a sequence $X_i = (X_i, F_i)$ with uniformly bounded areas converges to (X, F), if and only if it converges weakly, that is, if for all $h \in C^0(G_2M)$ we have

$$\int_{X_i} h \circ F_i \, J_{f_i} \to \int_X h \circ F \, J_f . \tag{3.37}$$

For instance, when M is a three-manifold, then G_2M, G_1M, and $T^1M/\{\pm v\}$ are isomorphic. (Here T^1M is the unit tangent bundle.) If Σ_i is a sequence of closed immersed surfaces in M converging to a closed surface Σ in the usual C^k topology, then we can think of each surface as being embedded in $T^1M/\{\pm v\} \equiv G_2M$ by mapping each point to plus-minus the unit normal vector, $\pm N$, to the surface. It follows easily that the surfaces with these inclusion maps converge in the varifold distance. More generally, if X is a compact surface and $f : X \to M$ is a $W^{1,2}$ map, where M is no longer assumed to be 3-dimensional, then we let $F : X \to G_2M$ be given by that $F(x)$ is the linear subspace $df(T_xX)$. (When M is 3-dimensional, then we may think of the image of this map as lying in $T^1M/\{\pm v\}$.) Strictly speaking, this is only defined on the measurable space where J_f is nonzero; we extend it arbitrarily to all of X since the corresponding Radon measure on G_2M given by $h \to \int_X h \circ F \, J_f$ is independent of the extension.

2. The Sobolev Inequality

We will next prove the Sobolev inequality of Michael and Simon, [**MiSi**]:

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[1] This is a corollary of the Stone-Weierstrass theorem; see corollary 35 on page 213 of [**R**].

Theorem 3.11. *There exists c_n depending only on n so that if $u \geq 0$ is a Lipschitz function with compact support on a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, then*

$$\left(\int_\Sigma u^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq c_n \int_\Sigma \left(|\nabla u| + u\,|H| \right) . \tag{3.38}$$

The above Sobolev inequality is, in fact, the extreme case of a family of inequalities. However, all of these inequalities follow from Theorem 3.11 and Hölder's inequality as we will see next.

Corollary 3.12. *There exists c_n depending only on n so that if $u \geq 0$ is a Lipschitz function with compact support on a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ and $p \in [1, n)$, then*

$$\|u\|_{L^{\frac{np}{n-p}}} \leq c_n \left(\frac{(n-1)p}{n-p} \|\nabla u\|_{L^p} + \|uH\|_{L^p} \right) . \tag{3.39}$$

Proof. Given $r > 1$, apply Theorem 3.11 to the function u^r to get

$$\|u\|^r_{L^{\frac{nr}{n-1}}} = \left(\int_\Sigma u^{\frac{nr}{n-1}} \right)^{\frac{n-1}{n}} \leq c_n \int_\Sigma \left(r\, u^{r-1} \,|\nabla u| + u^r\,|H| \right) . \tag{3.40}$$

Applying Hölder's inequality with exponents $\frac{nr}{(n-1)(r-1)}$ and $\frac{nr}{n+r-1}$ gives

$$\int_\Sigma u^{r-1}\,|\nabla u| \leq \left(\int_\Sigma u^{\frac{nr}{n-1}} \right)^{\frac{(n-1)(r-1)}{nr}} \left(\int_\Sigma |\nabla u|^{\frac{nr}{n+r-1}} \right)^{\frac{n+r-1}{nr}} ,$$
$$\int_\Sigma u^r\,|H| \leq \left(\int_\Sigma u^{\frac{nr}{n-1}} \right)^{\frac{(n-1)(r-1)}{nr}} \left(\int_\Sigma (u\,|H|)^{\frac{nr}{n+r-1}} \right)^{\frac{n+r-1}{nr}} .$$

Using these two bounds for the terms on the right-hand side of (3.40) and dividing through by

$$\left(\int_\Sigma u^{\frac{nr}{n-1}} \right)^{\frac{(n-1)(r-1)}{nr}} = \|u\|^{r-1}_{L^{\frac{nr}{n-1}}}$$

then gives

$$\|u\|_{L^{\frac{nr}{n-1}}} \leq c_n \left(r\,\|\nabla u\|_{L^{\frac{nr}{n+r-1}}} + \|uH\|_{L^{\frac{nr}{n+r-1}}} \right) .$$

Setting $p = \frac{nr}{n+r-1}$, so that $r = \frac{(n-1)p}{n-p}$ and $\frac{nr}{n-1} = \frac{np}{n-p}$, gives the corollary. □

The Michael-Simon Sobolev inequality was generalized by Hoffman and Spruck, [**HoSp**], to submanifolds of a general Riemannian manifold.

2.1. The proof of the Sobolev inequality. We will need the following covering lemma (theorem 3.3 in [**Si3**]):

Lemma 3.13. *If $\mathcal{B}$ is a family of closed balls in a metric space with*

$$\sup\{\mathrm{diam}(B) \mid B \in \mathcal{B}\} < \infty\,,$$

then there is a pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ so that

$$\bigcup_{B\in\mathcal{B}} \subset \bigcup_{B\in\mathcal{B}'} 5B\,,$$

where $5B$ is the ball with the same center as B but with 5 times the radius.

Proof. Set $d = \sup\{\mathrm{diam}(B) \mid B \in \mathcal{B}\}$ and subdivide $\mathcal{B}$ into disjoint collections $\mathcal{B}_1, \mathcal{B}_2, \ldots$ with

$$\mathcal{B}_j = \{B \in \mathcal{B} \mid \mathrm{diam}(B) \in (2^{-j}\, d, 2^{1-j}\, d)\}\,.$$

Let $\mathcal{B}_1'$ be any maximal pairwise disjoint subcollection of balls in $\mathcal{B}_1$. Next, let $\mathcal{B}_2'$ be a maximal pairwise disjoint subcollection of

$$\{B \in \mathcal{B}_2 \mid B \cap B' = \emptyset \text{ for every } B' \in \mathcal{B}_1'\}\,.$$

Continue inductively by letting $\mathcal{B}_j'$ be any maximal pairwise disjoint subcollection of balls in

$$\{B \in \mathcal{B}_j \mid B \cap B' = \emptyset \text{ for every } B' \in \bigcup_{i<j} \mathcal{B}_i'\}\,.$$

Now define $\mathcal{B}' = \bigcup_i \mathcal{B}_i'$. This is clearly pairwise disjoint.

Given $B \in \mathcal{B}$, then it must be contained in some $\mathcal{B}_j$. Clearly, we are done if $B \in \mathcal{B}_j'$, so we may assume that it is not. By maximality of the collection $\mathcal{B}_j'$, there must be some $i \leq j$ and a ball $B' \in \mathcal{B}_i$ with

$$B \cap B' \neq \emptyset\,.$$

The fact that $i \leq j$ implies that $\mathrm{diam}\, B \leq 2\, \mathrm{diam}\, B'$. It now follows from the triangle inequality that $B \subset 5B'$, completing the proof. □

We will follow Leon Simon's proof of the Sobolev inequality in [**Si3**]. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface and define $J(r)$ by

$$J(r) = \frac{1}{\omega_n\, r^n} \int_{B_r \cap \Sigma} h\,, \tag{3.41}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and h is a nonnegative function with compact support. The mean value inequality of Lemma 1.18 gives

$$r^{n+1}\, J'(r) \geq \frac{1}{\omega_n} \int_{B_r \cap \Sigma} \langle x, \nabla h + h\, H\, N\rangle \geq -\frac{r}{\omega_n} \int_{B_r \cap \Sigma} (|\nabla h| + h\, |H|)\,. \tag{3.42}$$

Integrating this gives for $0 < \sigma < \rho$ that

$$J(\sigma) \leq J(\rho) + \int_{\sigma}^{\rho} \left(r^{-n} \frac{1}{\omega_n} \int_{B_r \cap \Sigma} (|\nabla h| + h\,|H|) \right) dr\,. \tag{3.43}$$

The next ingredient is a calculus lemma (cf. 18.7 in [**Si3**]):

Lemma 3.14. *Suppose $f, g \geq 0$ are bounded, nondecreasing, with*

$$1 \leq \limsup_{\sigma \to 0} \frac{f(\sigma)}{\sigma^n}\,, \tag{3.44}$$

and for any $0 < \sigma < \rho$,

$$\frac{f(\sigma)}{\sigma^n} \leq \frac{f(\rho)}{\rho^n} + \int_0^{\rho} t^{-n}\, g(t)\, dt\,. \tag{3.45}$$

Set $f_\infty = \lim_{t \to \infty} f(t)$ and $\rho_0 = 2\,(f_\infty)^{1/n}$. Then there exists $\rho \in (0, \rho_0)$ with

$$f(5\rho) \leq \frac{5^n}{2}\, \rho_0\, g(\rho)\,. \tag{3.46}$$

Proof. We will argue by contradiction, so suppose instead that (3.46) fails for every $\rho \in (0, \rho_0)$. Combining this with (3.45) gives

$$\begin{aligned} \sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} &\leq \frac{f(\rho_0)}{\rho_0^n} + \int_0^{\rho_0} t^{-n}\, g(t)\, dt \\ &\leq \frac{f_\infty}{\rho_0^n} + \frac{2}{5^n\, \rho_0} \int_0^{\rho_0} t^{-n}\, f(5t)\, dt\,. \end{aligned} \tag{3.47}$$

To bound the second integral, make the change of variables $s = 5t$ to get

$$\begin{aligned} \frac{2}{5^n\, \rho_0} \int_0^{\rho_0} t^{-n}\, f(5t)\, dt &= \frac{2}{5\, \rho_0} \int_0^{5\,\rho_0} s^{-n}\, f(s)\, ds \\ &= \frac{2}{5\, \rho_0} \int_0^{\rho_0} s^{-n}\, f(s)\, ds + \frac{2}{5\, \rho_0} \int_{\rho_0}^{5\,\rho_0} s^{-n}\, f(s)\, ds \\ &\leq \frac{2}{5} \sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} + \frac{2}{5\,(n-1)}\, \rho_0^{-n}\, f_\infty\,. \end{aligned}$$

Substituting this back into (3.47) gives

$$\sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} \leq \frac{f_\infty}{\rho_0^n} + \frac{2}{5} \sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} + \frac{2}{5\,(n-1)} \frac{f_\infty}{\rho_0^n}\,,$$

and subtracting the second term on the right from both sides gives

$$\frac{3}{5} \sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} \leq \frac{5n-3}{5(n-1)} \frac{f_\infty}{\rho_0^n} = \frac{5n-3}{5(n-1)}\, 2^{-n}\,. \tag{3.48}$$

The last equality used the definition of ρ_0. Multiplying by 5/3 and noting that $\frac{5n-3}{5(n-1)} \leq 7/5$ since $n \geq 2$, we get

$$\sup_{\sigma \in (0,\rho_0)} \frac{f(\sigma)}{\sigma^n} \leq \frac{7}{3} 2^{-n} \leq \frac{7}{12} < 1, \tag{3.49}$$

but this contradicts (3.44) and, thus, proves the lemma. □

If $y \in \Sigma$ is a point with $h(y) \geq 1$, then (3.44) and (3.45) both hold with

$$f(\rho) = \frac{1}{\omega_n} \int_{B_\rho(y)\cap\Sigma} h,$$
$$g(\rho) = \frac{1}{\omega_n} \int_{B_\rho(y)\cap\Sigma} (|\nabla h| + h\,|H|).$$

(We use (3.43) to show that (3.45) holds; to get (3.44), note that $\lim_{\rho\to 0} f(\rho)$ is just $h(y)$ times the multiplicity of Σ at y.) With this choice of f, we have $f_\infty = \frac{1}{\omega_n} \int_\Sigma h < \infty$ (since h has compact support) so that

$$\rho_0 = 2\, f_\infty^{1/n} = 2 \left(\frac{1}{\omega_n} \int_\Sigma h \right)^{1/n}.$$

As a consequence, Lemma 3.14 gives a radius ρ_y with

$$\int_{B_{5\rho_y}(y)\cap\Sigma} h \leq \frac{5^n}{\omega_n^{1/n}} \left(\int_\Sigma h \right)^{1/n} \int_{B_{\rho_y}(y)\cap\Sigma} (|\nabla h| + h\,|H|). \tag{3.50}$$

Let $S \subset \Sigma$ be the set where $h \geq 1$:

$$S = \{ y \in \Sigma \,|\, h(y) \geq 1 \}.$$

Using the 5-times covering lemma, Lemma 3.13, we can choose disjoint balls B^j satisfying (3.50) and so that

$$S \subset \cup_j 5(B^j). \tag{3.51}$$

It follows that

$$\int_S h \leq \frac{5^n}{\omega_n^{1/n}} \left(\int_\Sigma h \right)^{1/n} \int_\Sigma (|\nabla h| + h\,|H|). \tag{3.52}$$

Fix some $\epsilon > 0$ and a monotone function $\gamma(t)$ with $\gamma(t) = 1$ for $t \geq \epsilon$ and $\gamma(t) = 0$ for $t \leq 0$. Given any $t > 0$, we set

$$A_t = \{ x \in \Sigma \,|\, u(x) > t \}. \tag{3.53}$$

Let μ be the volume measure on Σ. Applying (3.52) with $h(x) = \gamma(u(x) - t)$ gives

$$\mu(A_{t+\epsilon}) \leq \frac{5^n}{\omega_n^{1/n}} \, (\mu(A_t))^{1/n} \int_\Sigma (|\nabla u|\, \gamma'(u-t) + \gamma(u-t)\,|H|). \tag{3.54}$$

Multiplying through by $(t+\epsilon)^{\frac{1}{n-1}}$ gives

$$
\begin{aligned}
&(t+\epsilon)^{\frac{1}{n-1}}\,\mu(A_{t+\epsilon}) \\
&\qquad \le \frac{5^n}{\omega_n^{1/n}} \left((t+\epsilon)^{\frac{n}{n-1}}\,\mu(A_t) \right)^{1/n} \int_\Sigma \left(|\nabla u|\,\gamma'(u-t) + \gamma(u-t)\,|H| \right) \\
&\qquad \le \frac{5^n}{\omega_n^{1/n}} \left(\int_\Sigma (u+\epsilon)^{\frac{n}{n-1}} \right)^{1/n} \left(\int_\Sigma |\nabla u|\,\gamma'(u-t) + \int_{A_t} |H| \right), \qquad (3.55)
\end{aligned}
$$

where the last inequality used the trivial inequality $t\,\mu(\{v>t\}) \le \int v$ for any nonnegative function v.

The last lemma that we will need is a standard application of Fubini's theorem:

Lemma 3.15. *Suppose that ν is a measure on a space X, $f \ge 0$ is in $L^1(\nu)$, $A_t = \{x \,|\, f(x) > t\}$, and $\alpha > 0$. Then*

$$
\frac{1}{\alpha} \int f^\alpha \, d\nu = \int_0^\infty t^{\alpha-1}\, \nu(A_t)\, dt\,. \qquad (3.56)
$$

It follows that if $t_0 \ge 0$, then

$$
\frac{1}{\alpha} \int_{A_{t_0}} (f^\alpha - t_0^\alpha)\, d\nu = \int_{t_0}^\infty t^{\alpha-1}\, \nu(A_t)\, dt\,. \qquad (3.57)
$$

Proof. The lemma follows from applying Fubini's theorem to the product space $X \times [t_0, \infty)$. □

We can now complete the proof of the Sobolev inequality.

Proof of Theorem 3.11. Integrating (3.55) in t and applying Lemma 3.15 (with $\nu = \mu$, $t_0 = \epsilon$, and $\alpha - 1 = \frac{1}{n-1}$, so that $\alpha = \frac{n}{n-1}$) to the first term gives

$$
\begin{aligned}
&\frac{n-1}{n} \int_{A_\epsilon} \left(u^{\frac{n}{n-1}} - \epsilon^{\frac{n}{n-1}} \right) \\
&\le \frac{5^n}{\omega_n^{1/n}} \left(\int_\Sigma (u+\epsilon)^{\frac{n}{n-1}} \right)^{\frac{1}{n}} \int_0^\infty \left(\int_\Sigma |\nabla u|\,\gamma'(u-t) + \int_{A_t} |H| \right) dt \\
&= \frac{5^n}{\omega_n^{1/n}} \left(\int_\Sigma (u+\epsilon)^{\frac{n}{n-1}} \right)^{\frac{1}{n}} \int_\Sigma \left(|\nabla u| + u\,|H| \right), \qquad (3.58)
\end{aligned}
$$

where the equality used the fundamental theorem of calculus (and Fubini's theorem to change the order of integration) to evaluate the γ' term and

Lemma 3.15 (with $\nu = |H|\,\mu$, $t_0 = 0$, and $\alpha = 1$) to evaluate the last term. Letting ϵ go to zero, we get

$$\frac{n-1}{n}\int u^{\frac{n}{n-1}} \leq \frac{5^n}{\omega_n^{1/n}}\left(\int_\Sigma u^{\frac{n}{n-1}}\right)^{1/n}\int_\Sigma \left(|\nabla u| + u\,|H|\right)\,, \tag{3.59}$$

and this gives the theorem with

$$c_n = \frac{n}{n-1}\,\frac{5^n}{\omega_n^{1/n}}\,. \qquad \square$$

3. The Weak Bernstein-Type Theorem

We will refer to the following result as the weak Bernstein-type theorem.

Theorem 3.16 (Colding-Minicozzi, [**CM16**]). *If $\Sigma^k \subset \mathbb{R}^n$ is a k-rectifiable stationary varifold with (volume) density at least 1 almost everywhere and bounded from above by V_Σ, then Σ must be contained in some affine subspace of dimension at most $(k+1)\,\frac{k}{k-1}\,\mathrm{e}^8\,2^{k+4}\,\mathrm{V}_\Sigma$.*

The constant in Theorem 3.16 is not sharp, and is not the best constant which can be obtained from the arguments of [**CM16**]. However, the geometric dependence is sharp (namely, the bound is linear in V_Σ which is optimal).

Notice that Theorem 3.16 bounds the number of linearly independent coordinate functions on Σ in terms of its volume. When V_Σ is sufficiently small, Allard's theorem implies that Σ^k is planar and hence that there are only k independent coordinate functions. We note that it follows from Theorem 3.16 that the δ in the Allard regularity theorem is independent of n.

A calibration argument (see [**Fe**] or [**Mr**]), cf. (1.15) and (1.16), shows that complex submanifolds of $\mathbf{C}^n = \mathbb{R}^{2n}$ are absolutely area-minimizing (and hence minimal). Using this, it is easy to see that affine algebraic varieties are examples of stationary rectifiable varifolds with bounded density.

We saw earlier that the coordinate functions are weakly harmonic on Σ (Proposition 3.6; cf. Proposition 1.7). Theorem 3.16 will follow from a bound for the dimensions of certain spaces of harmonic functions on Σ, namely the so-called harmonic functions of polynomial growth.

Definition 3.17 (Harmonic Functions of Polynomial Growth). We will define $\mathcal{H}_d(\Sigma)$ to be the linear space of harmonic functions with polynomial growth of order at most d. That is, $u \in \mathcal{H}_d(\Sigma)$ if u is harmonic and there exists some $C < \infty$ so that $|u(x)| \leq C(1 + |x|^d)$.

With this definition, the coordinate functions x_i are in $\mathcal{H}_1(\Sigma)$. For example, on the catenoid $\Sigma_c^2 \subset \mathbb{R}^3$ centered on the x_3-axis, the function x_3 grows slower than any power of the intrinsic distance; however, it is not in $\mathcal{H}_d(\Sigma_c^2, \mathbb{R})$ for any $d < 1$.

Theorem 3.18 (Colding-Minicozzi, [**CM16**]). *Let Σ^k be a stationary k-rectifiable varifold with density at least 1 almost everywhere and bounded from above by $\mathrm{V}_\Sigma < \infty$. For any $d \geq 1$,*

$$\dim \mathcal{H}_d(\Sigma) \leq C \, \mathrm{V}_\Sigma \, d^{k-1} \, , \tag{3.60}$$

where $C = (k+1) \, \frac{k}{k-1} \, \mathrm{e}^8 \, 2^{k+4}$.

The proofs of the finite dimensionality results here consist of the following two independent steps. First, we will reduce the problem to bounding the number of $L^2(B_r \cap \Sigma)$-orthonormal harmonic functions on Σ with a uniform bound on the $L^2(B_{\Omega r} \cap \Sigma)$-norm for $r > 0$ and $\Omega > 1$. Next, we will use the linearity of the space of harmonic functions to form a "Bergman kernel". That is, if $u_1, \ldots, u_{\mathcal{N}}$ are the harmonic functions, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |u_i(x)|^2 \, . \tag{3.61}$$

Combining some standard linear algebra with the mean value inequality, we obtain pointwise bounds for $K(x)$ depending on Ω and the $L^2(B_{\Omega r} \cap \Sigma)$-norm but independent of $\mathcal{N}$. Integrating this over $B_r \cap \Sigma$ gives a bound on $\mathcal{N}$ which depends on Ω.

To bound the dimension of $\mathcal{H}_d(\Sigma)$ polynomially in d, we choose Ω to be approximately $1 + \frac{1}{d}$. The resulting estimates are then polynomial in d where the degree depends on both the rate of blowup at the boundary of the mean value inequality and the regularity of the volume measure.

See [**CM16**] for related and more general results in this direction. For results on harmonic functions with polynomial growth in other contexts, see [**CM14**], [**CM15**], and the references therein. Finally, see Calle, [**Ca**], for a generalization to mean curvature flow.

4. General Constructions

In this section, we will study the growth and independence properties of spaces of functions following [**CM16**]. To begin with, the following lemma illustrates the usefulness of the assumption on the growth rates of functions.

Lemma 3.19. *Suppose that $f_1, \ldots, f_{2s}$ are nonnegative nondecreasing functions on $(0, \infty)$ such that none of the f_i vanishes identically and for some*

d_0, $K > 0$ and all i,

$$f_i(r) \leq K(r^{d_0} + 1)\,. \tag{3.62}$$

For all $\Omega > 1$, there exist s of these functions $f_{\alpha_1}, \dots, f_{\alpha_s}$ and infinitely many integers, $m \geq 1$, such that for $i = 1, \dots, s$

$$f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{2d_0} f_{\alpha_i}(\Omega^m)\,. \tag{3.63}$$

Proof. Since the functions are nondecreasing and none of them vanish identically, we may suppose that for some $R > 0$ and any $r > R$, $f_i(r) > 0$ for all i.

We will show that there are infinitely many m such that there is some rank s subset of $\{f_i\}$ (where the subset could vary with m) satisfying (3.63). This will suffice to prove the lemma; since there are only finitely many rank s subsets of the $2s$ functions, one of these rank s subsets must be repeated infinitely often.

For $r > R$, note that

$$g(r) = \prod_{i=1}^{2s} f_i(r) \leq K^{2s}(r^{d_0} + 1)^{2s}\,, \tag{3.64}$$

and g is a positive nondecreasing function. Assume that there are only finitely many $m \geq \frac{\log R}{\log \Omega}$ satisfying (3.63). Let $m_0 - 1$ be the largest such m; for all $j \geq 1$ we have that

$$\Omega^{2d_0(s+1)} g(\Omega^{m_0+j-1}) < g(\Omega^{m_0+j})\,. \tag{3.65}$$

Iterating this and applying (3.64) gives for any $j \geq 1$,

$$\Omega^{2d_0(s+1)j} g(\Omega^{m_0}) < g(\Omega^{m_0+j}) \leq \tilde{c}\left(\Omega^j\right)^{2sd_0}\,, \tag{3.66}$$

where $\tilde{c} = \tilde{c}(s, m_0, \Omega, K)$. Since $\Omega > 1$, taking j large yields the contradiction. $\square$

For $r > 0$, $B_r = \{|x| < r\} \subset \mathbb{R}^n$, and functions u and v, let

$$I_u(r) = \int_{B_r \cap \Sigma} u^2 \tag{3.67}$$

and

$$J_r(u, v) = \int_{B_r \cap \Sigma} u\,v\,. \tag{3.68}$$

Note that J_r is an inner product and $I(r)$ is the corresponding quadratic form. Furthermore, if $u \in \mathcal{H}_d(\Sigma)$, then $I_u(r) \leq C(1 + r^{2d+k})$.

Given a linearly independent set of functions in $\mathcal{H}_d(\Sigma)$, we will construct functions of one variable which reflect the growth and independence properties of this set.

We begin with two definitions. The first constructs the functions whose growth properties will be studied.

Definition 3.20 ($w_{i,r}$ and f_i). Suppose that $u_1, \dots, u_s$ are linearly independent functions. For each $r > 0$ we will now define an J_r-orthogonal spanning set $w_{i,r}$ and functions f_i. Set $w_{1,r} = w_1 = u_1$ and $f_1(r) = I_{w_1}(r)$. Define $w_{i,r}$ by requiring it to be orthogonal to $u_j|B_r$ for $j < i$ with respect to the inner product J_r and so that

$$u_i = \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j + w_{i,r}\,. \tag{3.69}$$

Note that $\lambda_{ij}(r)$ is not uniquely defined if the $u_i|B_r$ are linearly dependent. However, since the u_i are linearly independent on Σ, $\lambda_{ij}(r)$ will be uniquely defined for r sufficiently large.

In any case, the following quantity is well defined for all $r > 0$ (and, in fact, is positive for r sufficiently large)

$$f_i(r) = \int_{B_r} |w_{i,r}|^2\,. \tag{3.70}$$

In the next proposition, we will record some key properties of the functions f_i from Definition 3.20.

Proposition 3.21 (Properties of f_i, [**CM16**]). *If $u_1, \dots, u_s \in \mathcal{H}_d(\Sigma)$ are linearly independent, the f_i from Definition 3.20 have the following four properties: There exists a constant $K > 0$ (depending on the set $\{u_i\}$) such that for $i = 1, \dots, s$*

(3.71) $\quad f_i(r) \leq K(r^{2d+k} + 1)\,,$

(3.72) $\quad f_i$ *is a nondecreasing function,*

(3.73) $\quad f_i$ *is nonnegative and positive for r sufficiently large, and*

(3.74) $\quad f_i(r) = I_{w_{i,r}}(r)$ *and* $f_i(t) \leq I_{w_{i,r}}(t)$ *for* $t < r$.

Proof. Note first that $f_i(r) \leq I_{u_i}(r)$; thus we get (3.71). Furthermore, for $s < r$,

$$\begin{aligned} f_i(s) &= \int_{B_s} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(s)\, u_j|^2 = I_{w_{i,s}}(s) \\ &\leq \int_{B_s} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j|^2 = I_{w_{i,r}}(s) \\ &\leq \int_{B_r} |u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j|^2 = I_{w_{i,r}}(r) = f_i(r)\,, \end{aligned} \tag{3.75}$$

where the first inequality of (3.75) follows from the orthogonality of $w_{i,r}$ to u_j for $j < i$, and the second inequality of (3.75) follows from the monotonicity of I. From (3.75) and the linear independence of the u_i, we get (3.72) and (3.73). Finally, (3.75) also contains (3.74). □

In the following proposition, we will apply Lemma 3.19 to the functions f_i from Definition 3.20:

Proposition 3.22 (Colding-Minicozzi, [**CM16**]). *Suppose that $u_1, \dots, u_{2s} \in \mathcal{H}_d(M)$ are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exist $m \geq m_0$, an integer $\ell \geq \frac{1}{2}\Omega^{-4d-2k}\, s$, and functions $v_1, \dots, v_\ell$ in the linear span of the u_i such that for $i, j = 1, \dots, \ell$,*

$$J_{\Omega^{m+1}}(v_i, v_j) = \delta_{i,j} \tag{3.76}$$

and

$$\frac{1}{2}\Omega^{-4d-2k} \leq I_{v_i}(\Omega^m)\,. \tag{3.77}$$

Proof. By (3.71), (3.72), and (3.74) of Proposition 3.21, applying Lemma 3.19 to the f_i of Definition 3.20 implies that there exist $m \geq m_0$ and a subset $f_{\alpha_1}, \dots, f_{\alpha_s}$ such that for $i = 1, \dots, s$,

$$0 < f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{4d+2k} f_{\alpha_i}(\Omega^m)\,. \tag{3.78}$$

Let $w_{\alpha_i,\Omega^{m+1}}$, $i = 1, \dots, s$, be the corresponding functions in the linear span of the u_i as in Definition 3.20.

Consider the s-dimensional linear space spanned by the functions $w_{\alpha_i,\Omega^{m+1}}$ with inner product $J_{\Omega^{m+1}}$. On this space there is also the positive semidefinite bilinear form J_{Ω^m}. Let $v_1, \dots, v_s$ be an orthonormal basis for $J_{\Omega^{m+1}}$ which diagonalizes J_{Ω^m}. We will now evaluate the trace of J_{Ω^m} with respect to these two bases. First, with respect to the orthogonal basis $w_{\alpha_i,\Omega^{m+1}}$ we get by (3.74) and (3.78),

$$s\,\Omega^{-4d-2k} \leq \sum_{i=1}^{s} \frac{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^m)}{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^{m+1})}\,. \tag{3.79}$$

Since the trace is independent of the choice of basis, we get when evaluating this on the orthonormal basis v_i,

$$s\,\Omega^{-4d-2k} \leq \sum_{i=1}^{s} I_{v_i}(\Omega^m)\,. \tag{3.80}$$

Combining this with

$$0 \leq I_{v_i}(\Omega^m) \leq 1\,, \tag{3.81}$$

which follows from the monotonicity of I, we get that there exist at least $\ell \geq \frac{s}{2}\,\Omega^{-4d-2k}$ of the v_i such that for each of these

$$\frac{1}{2}\,\Omega^{-4d-2k} \leq I_{v_i}(\Omega^m) \leq I_{v_i}(\Omega^{m+1}) = 1\,. \tag{3.82}$$

This shows the proposition. □

5. Finite Dimensionality

In this section, we will show how to bound the dimension of the space of polynomial growth functions which satisfy a mean value inequality following [**CM16**]. The bound on the dimension will be polynomial in the rate of growth with the exponent determined by the boundary blowup of the mean value inequality.

Following [**CM16**], we will say Σ has the *ϵ-volume regularity property* if for $0 < \epsilon \leq 1$ and $1 \leq C_W < \infty$, given any $0 < \delta \leq \frac{1}{2}$ we get an $R_0 > 0$ such that for all $r \geq R_0$,

$$\mathrm{Vol}(B_r \setminus B_{(1-\delta)r} \cap \Sigma) \leq C_W\, \delta^{\epsilon}\, \mathrm{Vol}(B_r \cap \Sigma)\,. \tag{3.83}$$

This is sometimes also called *micro-doubling.*

For example, it follows immediately that any k-dimensional cone has the 1-volume regularity property. We will not use it here, but note that lemma 3.3 in [**CM16**] shows that the geodesic metric spaces with a volume doubling satisfy this volume regularity; see [**Ch**], [**NrPy**], [**NrTa**], and [**Te**] for related results and applications.

Stationary k-rectifiable varifolds with density bounded above and below have the 1-volume regularity property. To see this, let Σ be a stationary k-rectifiable varifold with density at least 1 almost everywhere and such that

$$\mathrm{V}_\Sigma \equiv \lim_{r \to \infty} \Theta_0(r) < \infty\,. \tag{3.84}$$

Given $0 < \delta \leq \frac{1}{2}$, choose R_0 such that for $R \geq R_0/2$,

$$\mathrm{V}_\Sigma - \Theta_0(R) < \delta\, \mathrm{V}_\Sigma\,, \tag{3.85}$$

so that for $R \geq R_0$ we have

$$\begin{aligned} \mathrm{Vol}(B_{(1-\delta)R} \cap \Sigma) &= \Theta_0((1-\delta)R)\ \mathrm{Vol}(B_1 \subset \mathbb{R}^k)\,(1-\delta)^k\, R^k \\ &\geq \mathrm{V}_\Sigma\ \mathrm{Vol}(B_1 \subset \mathbb{R}^k)\,(1-\delta)^{k+1}\, R^k\,. \end{aligned} \tag{3.86}$$

Consequently, for $R \geq R_0$,

$$\begin{aligned} \mathrm{Vol}(B_R \setminus B_{(1-\delta)R} \cap \Sigma) &\leq \mathrm{V}_\Sigma\ \mathrm{Vol}(B_1 \subset \mathbb{R}^k)\left(1 - (1-\delta)^{k+1}\right) R^k \\ &\leq 2\,(k+1)\,\delta\ \mathrm{Vol}(B_R \cap \Sigma)\,. \end{aligned} \tag{3.87}$$

The following proposition will be used in combination with the results of Section 4:

Proposition 3.23 (Colding-Minicozzi, [**CM16**]). *Let Σ^k be a stationary k-rectifiable varifold with density at least 1 almost everywhere and bounded above by $\mathrm{V}_\Sigma < \infty$. Suppose that $0 < a < 1$ is fixed, $r > 2\,R_0$, and $v_1, \ldots, v_{\mathcal{N}}$ are harmonic and J_r-orthonormal. Given any $d \geq 1$ such that for any $R \geq R_0$ (3.87) holds for any $\delta \geq \frac{1}{4d}$ and for all i,*

$$a \leq I_{v_i}((1-(2d)^{-1})r)\,; \tag{3.88}$$

then

$$\mathcal{N} \leq C\, d^{k-1}\,, \tag{3.89}$$

where $C = \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\,a^{-1}\,2^{k+1}$.

Proof. Since $d \geq 1$, we can choose a positive integer N with $d \leq N \leq 2d$. For each $x \in B_r$, set

$$K(x) = \sum_{i=1}^{\mathcal{N}} |v_i|^2(x)\,. \tag{3.90}$$

Note that, since each $v_i \in L^2_{loc}(M)$, $K(x)$ must be finite by the mean value inequality. By construction, $K(x)$ is the trace of the symmetric bilinear form

$$(v, w) \to \langle v, w\rangle(x) \tag{3.91}$$

for any v, w in the span of the v_i.

Recall that a symmetric matrix can always be diagonalized by an orthogonal change of basis. Therefore, given $x \in B_r$, we can choose a new orthonormal basis $\{w_i\}$ for the inner product space $(\mathrm{span}\{v_i\}, J_r)$ such that at most one of the w_i, say w_1, is nonvanishing at x. Using the invariance of the trace under orthogonal change of basis, we have that

$$K(x) = \sum_{i=1}^{\mathcal{N}} |w_i|^2(x) = w_1^2(x)\,. \tag{3.92}$$

Now, since each w_i has L^2-norm one on B_r, the mean value inequality gives for $0 < s \leq \frac{1}{2}$ and any $x \in B_{(1-s)r}$,

$$|w_i|^2(x) \leq \frac{1}{\mathrm{Vol}(B_{s\,r} \subset \mathbb{R}^k)} \int_{B_{s\,r}(x)} w_i^2 \leq \frac{\mathrm{V}_\Sigma\; s^{-k}}{\mathrm{Vol}(B_r \cap \Sigma)}\,. \tag{3.93}$$

Combining (3.92) and (3.93), we get

$$\mathrm{Vol}(B_r \cap \Sigma)\, K(x) = \mathrm{Vol}(B_r \cap \Sigma)\, |w_1|^2(x) \leq \mathrm{V}_\Sigma \left(\frac{j}{2d}\right)^{-k}\,, \tag{3.94}$$

for each $j = 1, \ldots, N$ and any $x \in B_{(1-j/(2d))r}$.

We break down the integral of K to get

$$\int_{B_{\left(1-\frac{1}{2d}\right)r}} K = \int_{B_{\left(1-\frac{N}{2d}\right)r}} K + \sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \,. \tag{3.95}$$

We will now bound each of the two terms in the right-hand side of (3.95). Since $d \le N \le 2d$, $B_{(1-N/(2d))r} \subset B_{\frac{r}{2}}$, by (3.94) we have

$$\int_{B_{\left(1-\frac{N}{2d}\right)r}} K \le \mathrm{V}_\Sigma \left(\frac{1}{2}\right)^{-k} \frac{\mathrm{Vol}(B_{\frac{r}{2}} \cap \Sigma)}{\mathrm{Vol}(B_r \cap \Sigma)} \le \mathrm{V}_\Sigma\, 2^k \,. \tag{3.96}$$

Bounding the integral of K on each annulus above in terms of its maximum, (3.94) yields

$$\begin{aligned} &\sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \\ &\le \mathrm{V}_\Sigma \sum_{j=1}^{N-1} \frac{\mathrm{Vol}(B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r} \cap \Sigma)}{\mathrm{Vol}(B_r \cap \Sigma)} \left(\frac{j}{2d}\right)^{-k} \\ &\le \mathrm{V}_\Sigma \sum_{j=1}^{N-1} 2(k+1) \left(\frac{1}{2d}\right) \left(\frac{j}{2d}\right)^{-k} , \end{aligned} \tag{3.97}$$

where the second inequality follows from (3.87) (the volume regularity property). Using the elementary inequality

$$\sum_{j=1}^{N-1} j^{-k} = 1 + \sum_{j=2}^{N-1} j^{-k} \le 1 + \int_1^\infty s^{-k}\, ds = \frac{k}{k-1} , \tag{3.98}$$

(3.97) implies that

$$\sum_{j=1}^{N-1} \int_{B_{\left(1-\frac{j}{2d}\right)r} \setminus B_{\left(1-\frac{j+1}{2d}\right)r}} K \le \mathrm{V}_\Sigma\, (k+1)\, \frac{k}{k-1}\, 2^k\, d^{k-1} \,. \tag{3.99}$$

Substituting (3.96) and (3.99) into (3.95) yields

$$\begin{aligned} \int_{B_{\left(1-\frac{1}{2d}\right)r}} K &\le \mathrm{V}_\Sigma\, 2^k \left(1 + (k+1)\, \frac{k}{k-1}\, d^{k-1}\right) \\ &\le \mathrm{V}_\Sigma\, 2^{k+1} (k+1)\, \frac{k}{k-1}\, d^{k-1} , \end{aligned} \tag{3.100}$$

since $d \ge 1$ and $k > 1$.

Combining (3.88) and (3.100), we get

$$\mathcal{N} \leq a^{-1} \int_{B_{\left(1-\frac{1}{2d}\right)r}} K \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\, a^{-1}\, 2^{k+1}\, d^{k-1}\,. \tag{3.101}$$ □

Theorem 3.18 now follows directly.

Proof of Theorem 3.18. Set $\Omega = (1 - 1/(2d))^{-1}$ and choose a positive integer m_0 such that $\Omega^{m_0} \geq R_0$. Suppose that $u_1, \dots, u_{2s} \in \mathcal{H}_d(\Sigma)$ are linearly independent. By Proposition 3.22, there exist $N \geq m_0$, an integer ℓ with

$$\frac{\mathrm{e}^{-4}}{2}\, s \leq \ell\,, \tag{3.102}$$

and J_r-orthonormal functions $f_1, \dots, f_\ell$ in the linear span of the u_i such that for $i = 1, \dots, \ell$,

$$\frac{\mathrm{e}^{-4}}{2} \leq I_{f_i}((1-(2d)^{-1})r)\,, \tag{3.103}$$

where we have set $r = \Omega^{N+1}$. Proposition 3.23 (with $a = \mathrm{e}^{-4}/2$) implies that

$$\ell \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\;\mathrm{e}^4\; 2^{k+2}\, d^{k-1}\,. \tag{3.104}$$

Combining (3.102) and (3.104),

$$\dim \mathcal{H}_d(\Sigma) \leq \mathrm{V}_\Sigma\,(k+1)\,\frac{k}{k-1}\;\mathrm{e}^8\; 2^{k+4}\, d^{k-1}\,. \tag{3.105}$$ □

We will close this chapter with a very brief discussion relating the problems in this chapter to results on minimal cones in Euclidean space.

We leave the proof of the following elementary lemma to the reader (see, for instance, [**CM11**] and [**CM15**] for more discussion):

Lemma 3.24 (cf. (2.179) in Chapter 2). *If g is an eigenfunction with eigenvalue λ on N^{k-1} and $p^2 + (k-2)p = \lambda$, then $r^p\, g$ is a harmonic function on $C(N)$. In fact, $r^p\, g \in \mathcal{H}_p(C(N))$.*

As a consequence of this lemma we see that spectral properties of N are equivalent to properties of harmonic functions which grow polynomially on the cone $C(N)$. From this point of view, Theorem 3.18 is roughly an analog of Weyl's asymptotic formula. See [**CM15**] for further developments on this point of view in a related context.

The spectral properties of spherical minimal submanifolds have been studied in their own right (see, for instance, Cheng, Li and Yau [**CgLiYa**], Choi and Wang [**CiWa**] (Theorem 7.11), and Li and Tian [**LiTi**]).

6. Bubble Convergence Implies Varifold Convergence

In this section, we will recall the notion of bubble convergence for a sequence of maps from $\mathbf{S}^2$ and then show that bubble convergence implies varifold convergence. This was proven in [**CM27**]. This fact will be used later when we prove the existence of minimal two-spheres by a min-max argument.

6.1. Bubble convergence. We will need a notion of convergence for a sequence v^j of $W^{1,2}$ maps to a collection $\{u_0, \dots, u_m\}$ of $W^{1,2}$ maps which is similar in spirit to the convergence in Gromov's compactness theorem for pseudo holomorphic curves, [**G**]. The notion that we will use is a slight weakening of the bubble tree convergence developed by Parker and Wolfson for J-holomorphic curves in [**PkW**] and used by Parker for harmonic maps in [**Pk**]. In our applications, the v^j's will be approximately harmonic while the limit maps u_i will be harmonic. We will need the next definition to make this precise.

S^+ and S^- will denote the northern and southern hemispheres in $\mathbf{S}^2$ and $p^+ = (0,0,1)$ and $p^- = (0,0,-1)$ the north and south poles.

Definition 3.25. *Given a ball $B_r(x) \subset \mathbf{S}^2$, the* conformal dilation *taking $B_r(x)$ to S^- is the composition of translation $x \to p^-$ followed by dilation of $\mathbf{S}^2$ about p^- taking $B_r(p^-)$ to S^-.*

The standard example of a conformal dilation comes from applying stereographic projection

$$\Pi : \mathbf{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 , \tag{3.106}$$

then dilating $\mathbb{R}^2$ by a positive $\lambda \neq 1$, and applying Π^{-1}.

In the definition below of convergence, the map u_0 will be the standard $W^{1,2}$-weak limit of the v^j's (see (B1)), while the other u_i's will arise as weak limits of the composition of the v^j's with a divergent sequence of conformal dilations of $\mathbf{S}^2$ (see (B2)). The condition (B3) guarantees that these limits all arise in genuinely distinct ways, and the condition (B4) means that together the u_i's account for all of the energy.

Definition 3.26. Bubble convergence. *We will say that a sequence v^j : $\mathbf{S}^2 \to M$ of $W^{1,2}$ maps converges to a collection of $W^{1,2}$ maps $u_0, \dots, u_m$: $\mathbf{S}^2 \to M$ if the following hold:*

(B1) *The v^j's converge weakly to u_0 in $W^{1,2}$ and there is a finite set $\mathcal{S}_0 = \{x_0^1, \dots, x_0^{k_0}\} \subset \mathbf{S}^2$ so that the v^j's converge strongly to u_0 in $W^{1,2}(K)$ for any compact $K \subset \mathbf{S}^2 \setminus \mathcal{S}_0$.*

(B2) *For each $i > 0$, we get a point $x_{\ell_i} \in \mathcal{S}_0$ and a sequence of balls $B_{r_{i,j}}(y_{i,j})$ with $y_{i,j} \to x_{\ell_i}$ and $r_{i,j} \to 0$. Furthermore, if $D_{i,j}$:*

$\mathbf{S}^2 \to \mathbf{S}^2$ *is the conformal dilation taking the southern hemisphere to* $B_{r_{i,j}}(y_{i,j})$*, then the maps* $v^j \circ D_{i,j}$ *converge to* u_i *as in (B1). Namely,* $v^j \circ D_{i,j} \to u_i$ *weakly in* $W^{1,2}(\mathbf{S}^2)$ *and there is a finite set* $\mathcal{S}_i$ *so that the* $v^j \circ D_{i,j}$*'s converge strongly in* $W^{1,2}(K)$ *for any compact* $K \subset \mathbf{S}^2 \setminus \mathcal{S}_i$.

(B3) *If* $i_1 \neq i_2$*, then* $\frac{r_{i_1,j}}{r_{i_2,j}} + \frac{r_{i_2,j}}{r_{i_1,j}} + \frac{|y_{i_1,j}-y_{i_2,j}|^2}{r_{i_1,j}\, r_{i_1,j}} \to \infty$.

(B4) *We get the energy equality* $\sum_{i=0}^{m} E(u_i) = \lim_{j\to\infty} E(v^j)$.

6.2. Two simple examples of bubble convergence. The simplest nontrivial example of bubble convergence is when each map $v^j = u \circ \Psi_j$ is the composition of a fixed harmonic map $u : \mathbf{S}^2 \to M$ with a divergent sequence of dilations $\Psi_j : \mathbf{S}^2 \to \mathbf{S}^2$. In this case, the v^j's converge to the constant map $u_0 = u(p_+)$ on each compact set of $\mathbf{S}^2 \setminus \{p_-\}$ and all of the energy concentrates at the single point $p_- = \mathcal{S}_0$. Composing the v^j's with the divergent sequence Ψ_j^{-1} of conformal dilations gives the limit $u_1 = u$.

For the second example, let

$$\Pi : \mathbf{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 \tag{3.107}$$

be stereographic projection and let $z = x + iy$ be complex coordinates on $\mathbb{R}^2 = \mathbf{C}$. If we set

$$f_j(z) = 1/(jz) + z = \frac{z^2 + 1/j}{z}, \tag{3.108}$$

then the maps

$$v^j = \Pi^{-1} \circ f_j \circ \Pi : \mathbf{S}^2 \to \mathbf{S}^2 \tag{3.109}$$

are conformal and, therefore, also harmonic. Since each v^j is a rational map of degree two, we have $\mathrm{E}(v^j) = \mathrm{Area}(v^j) = 8\pi$. Moreover, the v^j's converge away from 0 to the identity map which has energy 4π. The other 4π of energy disappears at 0 but can be accounted for by a map u_1 by composing with a divergent sequence of conformal dilations; u_1 must also have degree one. In this case, the conformal dilations take f_j to

$$\tilde{f}_j(z) = f_j(z/j) = 1/z + z/j \tag{3.110}$$

which converges to the conformal inversion about the circle of radius one.

6.3. Bubble convergence implies varifold convergence.

Proposition 3.27 (Colding-Minicozzi, [**CM27**]). *If a sequence* v^j *of* $W^{1,2}(\mathbf{S}^2, M)$ *maps bubble converges to a finite collection of smooth maps* $u_0, \ldots, u_m : \mathbf{S}^2 \to M$*, then it also varifold converges.*

Before getting to the proof, recall that a sequence of functions f_j is said to *converge in measure* to a function f if for all $\delta > 0$ the measure of

$$\{x \mid |f_j - f|(x) > \delta\} \tag{3.111}$$

goes to zero as $j \to \infty$; see [**R**], page 95. Clearly, L^1 convergence implies convergence in measure. Furthermore, if $f_j \to f$ in measure and h is uniformly continuous, then

$$h \circ f_j \to h \circ f$$

in measure. Finally, we will use the following general version of the dominated convergence theorem which combines theorem 17 on page 92 of [**R**] and proposition 20 on page 96 of [**R**]:

(DCT) If $f_j \to f$ in measure, $g_j \to g$ in L^1, and $|f_j| \leq g_j$, then $\int f_j \to \int f$.

We will also use that the map $\nabla u \to J_u$ is continuous as a map from L^2 to L^1 and, thus, $\text{Area}(u)$ is continuous with respect to $\text{E}(u)$. To be precise, if $u, v \in W^{1,2}(\mathbf{S}^2, M)$, then

$$|J_u - J_v| \leq \sqrt{2}\, |\nabla u - \nabla v|^{1/2} \max\{|\nabla u|^{3/2}, |\nabla v|^{3/2}\}\,. \tag{3.112}$$

This follows from the linear algebra fact[2] that if S and T are $N \times 2$ matrices, then

$$\left|\det\left(S^T S\right) - \det\left(T^T T\right)\right| \leq 2\, |T - S| \max\{|S|^3, |T|^3\}\,, \tag{3.113}$$

where $|S|^2$ is the sum of the squares of the entries of S and S^T is the transpose.

Proof of Proposition 3.27. For each v^j, we will let V^j denote the corresponding map to G_2M. Similarly, for each u_i, let U_i denote the corresponding map to G_2M.

It follows from (B1)-(B4) that we can choose $m+1$ sequences of domains $\Omega_0^j, \ldots, \Omega_m^j \subset \mathbf{S}^2$ that are pairwise disjoint for each j and so that for each $i = 0, \ldots, m$ applying $D_{i,j}^{-1}$ to Ω_i^j gives a sequence of domains converging to $\mathbf{S}^2 \setminus \mathcal{S}_i$ and accounts for all the energy, that is,

$$\lim_{j \to \infty} \int_{\mathbf{S}^2 \setminus (\bigcup_i \Omega_i^j)} |\nabla v^j|^2 = 0\,. \tag{3.114}$$

[2]Note that $|S^T T| \leq |S|\,|T|$, $|\text{Tr}\,(S^T T)| \leq |S|\,|T|$, and if X_t is a path of 2×2 matrices, then $\partial_t \det X_t = \text{Tr}\,(X_t^c\, \partial_t X_t)$ where X_t^c is the cofactor matrix given by swapping diagonal entries and multiplying off-diagonals by -1. Applying this to $X_t = (S + t\,(T - S))^T\,(S + t\,(T - S))$ and using the mean value theorem gives (3.113).

By (3.114), the proposition follows from showing for each i and any h in $C^0(G_2M)$ that

$$\begin{aligned}\int_{\mathbf{S}^2} h \circ U_i\, J_{u_i} &= \lim_{j\to\infty} \int_{\Omega_i^j} h \circ V^j\, J_{v^j} \\ &= \lim_{j\to\infty} \int_{D_{i,j}^{-1}(\Omega_i^j)} h \circ V^j \circ D_{i,j}\, J_{(v^j \circ D_{i,j})}\,,\end{aligned} \tag{3.115}$$

where the last equality is simply the change of variables formula for integration.

To simplify notation in the proof of (3.115), for each i and j, let v_i^j denote the restriction of $v^j \circ D_{i,j}$ to $D_{i,j}^{-1}\left(\Omega_i^j\right)$ and let V_i^j denote the corresponding map to G_2M.

Observe that $J_{v_i^j} \to J_{u_i}$ in $L^1(\mathbf{S}^2)$ by (3.112). Given $\epsilon > 0$ and i, define

$$\Omega_\epsilon^i = \{J_{u_i} \geq \epsilon\}\,.$$

Since h is bounded and $J_{v_i^j} \to J_{u_i}$ in $L^1(\mathbf{S}^2)$, (3.115) would follow from

$$\lim_{j\to\infty} \int_{\Omega_\epsilon^i} h \circ V_i^j\, J_{v_i^j} = \int_{\Omega_\epsilon^i} h \circ U_i\, J_{u_i}\,. \tag{3.116}$$

However, given any $\delta > 0$, $W^{1,2}$ convergence implies that the measure of

$$\{x \in \Omega_\epsilon^i \mid J_{v_i^j} \geq \frac{\epsilon}{2} \text{ and } |V_i^j - U_i| \geq \delta\} \tag{3.117}$$

goes to zero as $j \to \infty$. Since L^1 convergence of Jacobians implies that the measure of $\{x \in \Omega_\epsilon^i \mid J_{v_i^j} < \frac{\epsilon}{2}\}$ goes to zero, it follows that the maps V_i^j converge in measure to U_i on Ω_ϵ^i. Therefore, the $h \circ V_i^j$'s converge in measure to $h \circ U_i$ on Ω_ϵ^i. Consequently, the general version of the dominated convergence theorem (DCT) gives (3.116) and, thus, also (3.115). □

Chapter 4

Existence Results

This chapter, focuses on the solution to the classical Plateau problem for maps from surfaces. There is a close connection between energy and area in dimension two and the main issue is to understand the lack of compactness, called "bubbling", for maps with bounded energy. The first three sections cover the basic existence results for the Dirichlet and Plateau problems for maps from disks, while the fourth section discusses branch points. After that, we turn to the existence of harmonic maps from the two-sphere, following the approach by Sacks and Uhlenbeck, [**SaUh**], of first minimizing a perturbed energy functional and then taking the limit as the perturbation goes to zero.

1. The Plateau Problem

The following fundamental existence problem for minimal surfaces is known as the Plateau problem:

> Given a closed curve Γ, find a minimal surface with boundary Γ.

This problem was first formulated by Lagrange in 1760 and was studied extensively by the Belgian physicist Plateau in the 19th century. This question has led to many significant developments in geometry and partial differential equations, including:

- Lebesgue's 1902 thesis where he constructed the Lebesgue integral in an effort to solve the Plateau problem.
- Early work on existence and regularity theory for partial differential equations of Courant, Morrey, and others.

- The regularity theory of De Giorgi, Reifenberg, and others.
- The development of geometric measure theory by Federer, Fleming, Almgren, and others.

There are various solutions to this problem depending on the exact definition of a surface (parameterized disk, integral current, $\mathbf{Z}_2$ current, or rectifiable varifold). We shall consider the version of the Plateau problem for parameterized disks; this was solved independently by J. Douglas [**Do**] and T. Rado [**Ra1**].

Theorem 4.1. *Given a piecewise C^1 closed Jordan curve $\Gamma \subset \mathbb{R}^3$, there exists a map $u : D \subset \mathbb{R}^2$ to $\mathbb{R}^3$ so that*

(1) *$u : \partial D \to \Gamma$ is monotone*[1] *and onto.*

(2) *$u \in C^0(\overline{D}) \cap W^{1,2}(D)$ and is C^∞ on D.*

(3) *The image of u minimizes area among all maps from disks with boundary Γ.*

The generalization to Riemannian manifolds is due to C. B. Morrey [**Mo1**].

The most natural approach to this problem would be to take a sequence of mappings whose areas are going to the infimum and attempt to extract a convergent subsequence. There are two serious difficulties with this.

First, since the area only depends on the image and not the parameterization, the noncompactness of the diffeomorphism group of the disk is a major problem. Namely, let $\phi_k : D \to D$ be a noncompact sequence of diffeomorphisms of the disk and fix $u : D \to \mathbb{R}^3$. The sequence of maps $u(\phi_k)$ has the same image but does not converge.

The second difficulty is that a bound on the area of a map does not give much control on the map. Since long thin tubes can have arbitrarily small area, it is possible to construct a sequence of surfaces in $\mathbb{R}^3$ with boundary $\partial D \subset \mathbb{R}^2 \subset \mathbb{R}^3$ with area close to π whose closure is all of $\mathbb{R}^3$. Namely, we can do the following:

Example 4.2. Let $T(x, y, h, k)$ denote the cylindrical "tentacle" centered at (x, y) of height h and width k (with the top closed off with a disk). This has excess area $\pi\, h\, k$. For each positive integer ℓ, let Σ_ℓ be the topological disk with the tentacles $T(\alpha/2^\ell, \beta/2^\ell, 1, 2^{-4\ell})$, $|\alpha|^2 + |\beta|^2 < 2^{2\ell}$, attached. The area of Σ_ℓ is less than $\pi + 2^{2\ell}(\pi\, 2^{-3\ell}) = \pi(1 + 2^{-\ell})$, and hence Σ_ℓ is a minimizing sequence. However, it is obvious that this sequence of surfaces does not converge.

[1] Monotone is defined below.

Instead of minimizing area, we will minimize energy (the L^2-norm of the differential of the map). Then we will show that the energy minimizer both minimizes area and finds a good parameterization.

Before giving a proof of this existence result, we shall need a few preliminaries.

$W^{1,2} = W^{1,2}(D)$ will denote the Sobolev space of functions f such that f and $|\nabla f|$ are square integrable on the disk D, and the $W^{1,2}$-norm of f is

$$|f|^2_{W^{1,2}} = \int_D |f|^2 + \int_D |\nabla f|^2 . \tag{4.1}$$

We set $C_0^\infty(D)$ equal to the space of smooth functions with compact support on D, and then let $W_0^{1,2}(D) \subset W^{1,2}(D)$ denote the closure of $C_0^\infty(D)$ with respect to the $W^{1,2}$-norm.

Let (x,y) be coordinates on $\mathbb{R}^2$ and suppose that $u = (u^1, u^2, u^3)$ is a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$.

If u is in $W^{1,2}(D)$, the energy is defined by

$$\mathrm{E}(u) = \frac{1}{2}\int_D |\nabla u|^2 \, dx\, dy = \frac{1}{2}\int_D \left(|u_x|^2 + |u_y|^2\right) \, dx\, dy , \tag{4.2}$$

where ∇ is the Euclidean gradient. The area (or two-dimensional Hausdorff measure) of the image of u is

$$\mathrm{Area}(u) = \int_D \left(|u_x|^2 \, |u_y|^2 - \langle u_x, u_y\rangle^2\right)^{\frac{1}{2}} \, dx\, dy . \tag{4.3}$$

We then have that

$$\mathrm{Area}(u) \leq \mathrm{E}(u) , \tag{4.4}$$

with equality if and only if $\langle u_x, u_y\rangle$ and $|u_x|^2 - |u_y|^2$ are zero (as L^1 functions). In the case of equality, we say that u is *almost conformal*. If u is an almost conformal immersion, then u is conformal.

One of the key advantages of working in two dimensions is the existence of isothermal coordinates. Namely, given u as above, then there exists a diffeomorphism $\phi : D \to D$ such that $u(\phi) : D \to \mathbb{R}^3$ is almost conformal (see Morrey [**Mo2**] or Chern [**Ch2**] for a proof). Since the area depends only on the image (and not the parameterization), we have $\mathrm{Area}(u(\phi)) = \mathrm{Area}(u)$.

Remark 4.3. In fact, in theorem 3 of [**Mo2**], Morrey showed that if ds^2 is a bounded piecewise smooth metric on D, then there exists a homeomorphism $\phi : D \to D$ with $\phi, |\nabla \phi|, |\nabla^2 \phi|$ in L^2 and such that ϕ is almost conformal. This stronger result will be applied later in Theorem 6.20.

Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1 closed Jordan curve. A map $f : \partial D \to \Gamma$ is said to be *monotone* if the inverse image of every connected set is connected. We define the class

$$X_\Gamma = \{\psi : D \to \mathbb{R}^3 \mid \psi \in C^0(\overline{D}) \cap W^{1,2}(D) \text{ and } \psi|_{\partial D} : \partial D \to \Gamma \text{ is monotone and onto}\}.$$

Define

$$A_\Gamma = \inf_{\psi \in X_\Gamma} \operatorname{Area}(\psi) \text{ and } E_\Gamma = \inf_{\psi \in X_\Gamma} \mathrm{E}(\psi). \tag{4.5}$$

The existence of isothermal coordinates gives the following lemma:

Lemma 4.4. $A_\Gamma = E_\Gamma$.

Proof. It follows immediately from (4.4) that $A_\Gamma \leq E_\Gamma$.

The other direction follows from the existence of isothermal coordinates (see Remark 4.3), although some care needs to be taken at the points where the mappings fail to be immersions. Given $\epsilon > 0$, choose some $u \in X_\Gamma$ with

$$\operatorname{Area}(u) = \int_D \sqrt{\det g_{ij}} < A_\Gamma + \epsilon/2\,,$$

where $g_{ij} = \langle u_i, u_j \rangle$ is the (possibly degenerate) pullback metric on D and

$$\operatorname{Tr} g = |\nabla u|^2\,.$$

The existence of isothermal coordinates requires that the metric be non-degenerate. To deal with this, define the perturbed map $u^s : D \to \mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ by

$$u^s(x, y) = (u(x, y), s\,x, s\,y)\,.$$

The pullback metric for u^s is

$$\tilde{g}_{ij} = g_{ij} + s^2\,\delta_{ij}\,.$$

For future reference, we record that

$$\begin{aligned} \det \tilde{g} &= \det\left(g_{ij} + s^2\,\delta_{ij}\right) = \det g + s^2\,(\operatorname{Tr} g) + s^4 = \det g + s^2\,|\nabla u|^2 + s^4 \\ &\leq \left(\sqrt{\det g} + s\,|\nabla u| + s^2\right)^2 . \end{aligned} \tag{4.6}$$

The existence of isothermal coordinates (see [**Mo2**] or [**Ch2**]) gives a conformal diffeomorphism $\phi : (D, \delta_{ij}) \to (D, \tilde{g})$, so we conclude that $u^s \circ \phi$ is conformal. By construction, we have

$$\mathrm{E}(u \circ \phi) \leq \mathrm{E}(u^s \circ \phi) = \operatorname{Area}(u^s \circ \phi) = \operatorname{Area}(u^s)\,,$$

where the first equality uses conformality and the last uses that ϕ is a diffeomorphism so does not change the area of the image.

We will show next that we can make the area of u^s as close as we like to the area of u by taking s small. Namely, (4.6) gives

$$\begin{aligned}\text{Area}\,(u^s) = \int_D \sqrt{\det \tilde{g}} &\leq \int_D \left(\sqrt{\det g} + s\,|\nabla u| + s^2\right) \\ &\leq \text{Area}(u) + \pi\, s^2 + s \int |\nabla u|\,.\end{aligned}$$

Since $u \in W^{1,2}$, we can choose s to make this less than $\text{Area}(u) + \epsilon$. Putting things together, we get that

$$E_\Gamma \leq \text{E}(u \circ \phi) \leq \text{Area}(u) + \epsilon < A_\Gamma + 2\,\epsilon\,,$$

which proves the lemma since $\epsilon > 0$ is arbitrary. □

We shall prove Theorem 4.1 in two steps. First, we prove that for each parameterization of the boundary curve Γ there is an energy-minimizing map from the disk with these boundary values. Each such map is harmonic. Since the target is Euclidean, this means that the components u^i are harmonic functions. This is, of course, the classical Dirichlet problem. Second, we will minimize energy over the possible parameterizations of the boundary to obtain *the* energy minimizer which will also solve the Plateau problem. Of course, to extract a limit here we will need uniform estimates for the solutions in the sequence. The key estimate for doing this is the Courant-Lebesgue lemma below.

2. The Dirichlet Problem

We will use the following version of the solution to the well-known Dirichlet problem for harmonic maps:

Proposition 4.5. *If $w \in C^0(\overline{D}) \cap W^{1,2}(D)$, then there is a unique (energy-minimizing) harmonic $u \in C^0(\overline{D}) \cap W^{1,2}(D) \cap C^\infty(D)$ with $u = w$ on ∂D.*

The rest of this section will deal with the proof of Proposition 4.5. There are a number of different possible approaches (using Fourier series, the Poisson kernel, the Perron process, etc.). We will use a variational approach, where we take a sequence of functions whose energy goes to the infimum and obtain a limiting energy minimizer.

2.1. Interior regularity of weakly harmonic functions. Before proving the proposition, we need some preliminaries. We shall use the following weak compactness result for $W^{1,2}$ functions (see theorem 7.22 of [**GiTr**] for a proof):

Lemma 4.6 (Rellich Compactness). *If u_k is a sequence in $W^{1,2}(D)$ with*

$$\sup_k \|u_k\|_{W^{1,2}} < \infty ,$$

then there is a subsequence $u_{k'}$ and $u \in W^{1,2}(D)$ so that $u_{k'} \to u$ strongly in L^2, $\nabla u_{k'} \to \nabla u$ weakly in L^2, and

$$\int_D |\nabla u|^2 \leq \liminf \int_D |\nabla u_k|^2 . \tag{4.7}$$

Moreover, if each u_k is in $W_0^{1,2}$, then so is u.

The last part of Lemma 4.6 is a special case of Sobolev space trace theory; see section 5.5 of [**Ev**] for more on this.

We will also need the following Dirichlet Poincaré inequality for the disk:

Lemma 4.7 (Dirichlet Poincaré Inequality). *There exists $C < \infty$ such that if $u \in W_0^{1,2}(D)$, then*

$$\int_D u^2 \leq C \int_D |\nabla u|^2 . \tag{4.8}$$

Proof. By density, we may assume that u is smooth and has compact support in D. The Sobolev inequality (see Chapter 3) gives a constant c so that

$$\int_D u^2 \leq c \left(\int_D |\nabla u| \right)^2 \leq c\pi \int_D |\nabla u|^2 ,$$

where the second inequality is Cauchy-Schwarz. □

A function $v \in W^{1,2}(D)$ is said to be weakly harmonic if for every smooth ψ with compact support in D,

$$\int_D \langle \nabla v, \nabla \psi \rangle \, dx \, dy = 0 . \tag{4.9}$$

It follows immediately from integration by parts that smooth harmonic functions are weakly harmonic.

We shall also need the following elementary regularity result for harmonic functions on the disk:

Lemma 4.8 (Weyl's Lemma). *If $v : D \to \mathbb{R}$ is in $W^{1,2}$ and is weakly harmonic, then v is C^∞ on the interior of D.*

Of course, this result follows from standard regularity theory (see, for instance, theorem 2.10 of [**GiTr**]), but we will give the proof because of its simplicity. The proof relies on convolution with an approximate identity, so we briefly recall this technique.

Let $\psi : [0,1] \to \mathbb{R}$ be a smooth nonnegative monotone nonincreasing function, that is constant on $[0, \frac{1}{3}]$, has support on $[0, \frac{2}{3}]$, and has

$$2\,\pi \int_0^1 \psi(t)\, t\, dt = 1\,. \tag{4.10}$$

Define $\phi_t : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi_t(x) = t^{-2}\, \psi(|x|/t)\,. \tag{4.11}$$

Hence, ϕ_t is a nonnegative smooth radially symmetric which is supported on D_t and has total integral one.

Fix some $t < 1$ for the moment and let $v_t : D_{1-t} \to \mathbb{R}$ be the convolution of v and ϕ_t. That is, given y with $|y| < (1-t)$ we have

$$v_t(y) = \int_{\mathbb{R}^2} v(y+x)\, \phi_t(x)\, dx\,. \tag{4.12}$$

This integral is well defined since ϕ_t is supported on a ball of radius t. The importance of this construction is that v_t is smooth. To see this, we do a change of variables $z = y + x$ and write (4.12) as

$$v_t(y) = \int_{\mathbb{R}^2} v(z)\, \phi_t(z-y)\, dz\,. \tag{4.13}$$

We can now differentiate (4.13) under the integral sign and the smoothness of v_t follows from the smoothness of ϕ_t (see, for instance, section 7.2 of [**GiTr**] for further discussion).

Proof of Lemma 4.8. Since v_t is smooth, the lemma will follow once we show that $v_t = v$.

To see this, we use the radial symmetry of ϕ_t and the mean value property for weakly harmonic functions which follows easily from Proposition 1.15. That is, Proposition 1.15 gives for a weakly harmonic function v and any r,

$$\int_{\theta=0}^{2\pi} v(y + (r\,\cos\theta, r\,\sin\theta))\, d\theta = 2\,\pi\, v(y)\,. \tag{4.14}$$

Writing (4.12) in polar coordinates, we get

$$\begin{aligned} v_t(y) &= \int_{r=0}^{t} \int_{\theta=0}^{2\pi} v(y + (r\,\cos\theta, r\,\sin\theta))\, \phi_t(r\,\cos\theta, r\,\sin\theta)\, r\, d\theta\, dr \\ &= \int_{r=0}^{t} t^{-2}\, \psi\left(\frac{r}{t}\right) \int_{\theta=0}^{2\pi} v(y + (r\,\cos\theta, r\,\sin\theta))\, r\, d\theta\, dr \qquad (4.15) \\ &= 2\pi\, v(y)\, t^{-2} \int_{r=0}^{t} \psi\left(\frac{r}{t}\right) r\, dr = v(y)\,. \end{aligned}$$

□

Note that the argument actually gives interior estimates for v and all of its derivatives. It is easy to see that the harmonic function is real analytic.

Corollary 4.9. *If $v : D \to \mathbb{R}$ is a smooth harmonic function, then v is real analytic in D.*

Proof. The complex-valued function $v = u_x - i\,u_y$ is smooth and we compute that

$$v_x = u_{x\,x} - i\,u_{y\,x}\,, \tag{4.16}$$

$$v_y = u_{x\,y} - i\,u_{y\,y}\,. \tag{4.17}$$

Equations (4.16) and (4.17) imply that v satisfies the Cauchy-Riemann equations and is hence holomorphic. Therefore the real and imaginary parts of v, namely u_x and u_y, are real analytic. Integrating this shows that u is itself real analytic. □

2.2. Boundary regularity. The previous subsection established interior regularity of weakly harmonic functions. We show next that harmonic functions are continuous on the closed disk when the boundary values are continuous. This was first proven by H.A. Schwarz and was important in the proof of the Schwarz reflection principle. For simplicity, we restrict to the simple case of the disk in $\mathbb{R}^2$.

Lemma 4.10. *Suppose that $w \in C^0(\overline{D}) \cap W^{1,2}(D)$. If u is weakly harmonic on D and $(u - w) \in W_0^{1,2}(D)$, then $u \in C^0(\overline{D})$.*

Proof. We have already shown interior regularity, so it remains only to show that v is continuous at each point of ∂D. By symmetry of ∂D, it suffices to show continuity at $x = 1, y = 0$. We will do this using the notion of a barrier. The function $(1 - x)$ is called a *barrier* at $(1, 0) \in \partial D$ because it satisfies:

- $(1 - x)$ vanishes at $(1, 0)$ and is positive in $\overline{D} \setminus (1, 0)$.
- $(1 - x)$ is continuous on $\overline{D}$ and superharmonic[2] in D.

Fix any $\epsilon > 0$. By the continuity of w, there exists $\delta > 0$ so that

$$|w(z) - w(1,0)| < \epsilon \text{ for all } z \in \overline{D} \cap B_\delta(1,0)\,.$$

Since $(1 - x) > 0$ on $\overline{D} \setminus (1, 0)$ and $\delta > 0$, there exists k so that

$$k\,(1 - x)(z) > 2 \sup_{\overline{D}} |w| \text{ for all } z \notin \overline{D} \cap B_\delta(1,0)\,.$$

We now define continuous harmonic functions w^+ and w^- by

$$w^\pm = w(1,0) \pm (\epsilon + k\,(1 - x))\,,$$

[2] Of course, $(1 - x)$ is actually harmonic, but superharmonic is all that is needed.

so that on $\overline{D}$ (and, thus, on ∂D)

$$w^- \leq w \leq w^+ .$$

The maximum principle (see theorem 8.1 in [**GiTr**] for a version that allows the harmonic functions to be in $W^{1,2}$) gives that on $\overline{D}$ we have

$$w^- \leq u \leq w^+ . \tag{4.18}$$

Since $(1-x)$ is continuous and vanishes at $(1,0)$, we can choose $\delta_0 > 0$ (and less than δ) so that

$$k\,(1-x) < \epsilon \text{ on } \overline{D} \cap B_{\delta_0}(1,0) .$$

Finally, we conclude from (4.18) that

$$|u - u(1,0)| < 2\,\epsilon \text{ on } \overline{D} \cap B_{\delta_0}(1,0) .$$

Since $\epsilon > 0$ was arbitrary, this gives the desired continuity and, thus, completes the proof. □

2.3. The solution of the Dirichlet problem. We are now prepared to solve the Dirichlet problem by the direct method in the calculus of variations: we show that a minimizing sequence converges to a weakly harmonic function which is then regular.

Proof of Proposition 4.5. Let $W^{1,2}_w$ denote the space of functions in $f \in W^{1,2}$ such that $(f - w)$ is in $W^{1,2}_0$, i.e.,

$$W^{1,2}_w = \left\{ f \in W^{1,2} \,|\, (f-w) \in W^{1,2}_0 \right\}$$

and let E_w be the infimum of the energy

$$E_w = \inf \left\{ \int_D |\nabla u|^2 \,|\, u \in W^{1,2}_w \right\} .$$

We use the direct method to get an energy-minimizer. Namely, choose a minimizing sequence of functions $u_\ell \in W^{1,2}_w$ with

$$\int_D |\nabla u_\ell|^2 < E_w + \frac{1}{\ell} . \tag{4.19}$$

Integrating the parallelogram law

$$\left| \nabla \frac{(u_i - u_j)}{2} \right|^2 + \left| \nabla \frac{(u_i + u_j)}{2} \right|^2 = \frac{1}{2}\,|\nabla u_i|^2 + \frac{1}{2}\,|\nabla u_i|^2$$

over the disk D gives

$$\begin{aligned} \frac{1}{4}\int_D |\nabla (u_i - u_j)|^2 + \int_D \left| \nabla \frac{(u_i + u_j)}{2} \right|^2 &= \frac{1}{2}\int_D |\nabla u_i|^2 + \frac{1}{2}\int_D |\nabla u_i|^2 \\ &< E_w + \frac{1}{2}(1/i + 1/j) . \end{aligned} \tag{4.20}$$

The function $\frac{1}{2}(u_i + u_j)$ agrees with w on the boundary, so it has energy at least E_w and subtracting E_w from both sides gives

$$\frac{1}{4}\int_D |\nabla(u_i - u_j)|^2 < \frac{1}{2}(1/i + 1/j)\,.$$

Since $(u_i - u_j) \in W_0^{1,2}$, Lemma 4.7 implies that

$$\int_D (u_i - u_j)^2 \leq C \int_D |\nabla(u_i - u_j)|^2 \leq 2\,C\,(1/i + 1/j)\,. \tag{4.21}$$

We conclude that the sequence u_ℓ is Cauchy in $W^{1,2}$ and therefore converges strongly in $W^{1,2}$ to a function $v \in W^{1,2}$ with

$$\int_D |\nabla v|^2 \leq E_w\,.$$

The sequence of $W_0^{1,2}(D)$ functions $(u_\ell - w)$ converges to $(v - w)$, so we conclude (by the last part of Rellich's compactness, Lemma 4.6) that $(v - w) \in W_0^{1,2}(D)$. Hence, $v \in W_w^{1,2}$ and the definition of E_w implies that

$$\int_D |\nabla v|^2 \geq E_w\,.$$

We conclude that $\int_D |\nabla v|^2 = E_w$ and v is energy-minimizing. Uniqueness follows immediately from the strong convergence of any minimizing sequence, without having to pass to subsequences. (It will also follow immediately from the maximum principle once we show that v is harmonic.)

We show next that v is weakly harmonic. Since v is energy-minimizing, we have for any smooth ψ with compact support in D that

$$\mathrm{E}(v) = \frac{1}{2}\int_D |\nabla v|^2 \leq \frac{1}{2}\,\mathrm{E}(v + t\,\psi) = \frac{1}{2}\int_D \langle \nabla v + t\nabla\psi, \nabla v + t\nabla\psi\rangle\,.$$

Differentiating this at $t = 0$ gives

$$0 = \frac{d}{dt}\,\mathrm{E}(v + t\,\psi)|_{t=0} = \int_D \langle \nabla v, \nabla\psi\rangle\,. \tag{4.22}$$

Since this holds for any smooth ψ with compact support in D, we conclude that v is weakly harmonic. Hence, by Lemma 4.8, v is smooth on the interior of the disk. Finally, Lemma 4.10 gives that v is continuous on the closed disk $\bar{D}$, completing the proof. □

3. The Solution to the Plateau Problem

We will now apply the solution of the Dirichlet problem in the previous section to solve the Plateau problem by minimizing over the possible boundary parameterizations.

Given any point $p \in D$, for each $\rho > 0$ we define the set

$$C_\rho = \{q \in D \mid |p - q| = \rho\}.$$

Given a map $u : D \to \mathbb{R}^3$, let $d(C_\rho)$ be the diameter of the image of the curve C_ρ, and $L(C_\rho)$ to be the length of the image of the curve C_ρ.

Lemma 4.11 (Courant-Lebesgue lemma). *If $u : D \to \mathbb{R}^3$, $u \in C^0(\overline{D}) \cap W^{1,2}(D)$ and $\mathrm{E}(u) \leq K/2$, then, for every $\delta < 1$, there exists $\rho \in [\delta, \sqrt{\delta}]$ with*

$$(d(C_\rho))^2 \leq 2\pi\epsilon_\delta, \tag{4.23}$$

where $\epsilon_\delta = \frac{4\pi K}{-\log\delta}$; in particular, $\epsilon_\delta \to 0$ as $\delta \to 0$.

Proof. By the usual density and regularization arguments (see, for instance, chapter 5 of [**Ev**]), it suffices to prove the lemma assuming that u is C^1. In this case, we will bound $L(C_\rho)$ and this immediately implies a bound on $d(C_\rho)$.

Define the quantity $p(r)$ by

$$p(r) = r \int_{C_r} |\nabla u|^2 \, ds\,, \tag{4.24}$$

where ds is arclength measure on C_r. We have

$$\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r) = \int_\delta^{\sqrt{\delta}} p(r)\, \frac{dr}{r} \leq \int_D |\nabla u|^2 \, dx\, dy \leq K\,. \tag{4.25}$$

Consequently, by the mean value theorem in one-variable calculus, there exists some ρ between δ and $\sqrt{\delta}$ such that

$$p(\rho) \leq \frac{\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r)}{\int_\delta^{\sqrt{\delta}} d(\log r)} \leq \frac{2K}{-\log\delta}\,. \tag{4.26}$$

Given any $r \in [\delta, \sqrt{\delta}]$, the Cauchy-Schwarz inequality gives

$$(L(C_r))^2 \leq \left(\int_{C_r} |\nabla u| \, ds\right)^2 \leq 2\pi\, p(r)\,. \tag{4.27}$$

Combining (4.26) and (4.27) gives $(L(C_\rho))^2 \leq \frac{4\pi K}{-\log\delta}$, as claimed. ☐

There is one final difficulty to overcome: the noncompactness of the conformal group of D. Recall that a diffeomorphism $\phi : D \to D$ which takes the boundary to the boundary is conformal if it preserves angles. This is equivalent to $\langle\phi_x, \phi_y\rangle = 0$ and $|\phi_x|^2 = |\phi_y|^2$. The noncompactness of the conformal group is a problem since energy is conformally invariant in dimension two.

Lemma 4.12. *If $u \in W^{1,2}(D)$ and $\phi : D \to D$ is a conformal diffeomorphism, then $E(u) = E(u(\phi))$.*

Proof. Let g_{ij} denote the pullback under ϕ of the Euclidean metric δ_{ij}. The conformality of ϕ gives that

$$g_{ij} = \langle \phi_i, \phi_j \rangle = \frac{1}{2} |\nabla \phi|^2 \delta_{ij} \,. \tag{4.28}$$

Metrics which are related by a scalar factor in this way are said to be conformal. Next, we will observe that energy is invariant under conformal changes of metric. Suppose that $g_{ij} = \lambda^2 \, \delta_{ij}$. We have

$$\begin{aligned} \mathrm{E}_g(u) &= \frac{1}{2} \int_D g^{ij} u_i u_j \, (\det g_{ij})^{\frac{1}{2}} \, dx \, dy \\ &= \frac{1}{2} \int_D \lambda^{-2} \, |\nabla u|^2 \, \lambda^2 \, dx \, dy = \mathrm{E}(u) \,. \end{aligned} \tag{4.29}$$

□

We will use that the conformal group of D is the group of linear fractional transformations. This group acts triply-transitively on the boundary, that is, given two triples of distinct points on the boundary, there exists a linear fractional transformation taking one to the other. However, the energy and area are both invariant under conformal reparameterizations of the domain; hence, we need to mod out this action.

For the remainder of this section, we fix three distinct points q_ℓ in Γ and three distinct points p_ℓ in ∂D. We have shown the following:

Lemma 4.13. *If $u : D \to \mathbb{R}^3$ is in $C^0(\overline{D}) \cap W^{1,2}(D)$, $u : \partial D \to \Gamma$ is monotone and onto, then there is a linear fractional transformation $\phi : D \to D$ so that $u \circ \phi$ satisfies:*

(1) *$E(u) = E(u \circ \phi)$.*
(2) *$u \circ \phi \in C^0(\overline{D}) \cap W^{1,2}(D)$.*
(3) *$u : \partial D \to \Gamma$ is monotone and onto.*
(4) *For each $i = 1, 2, 3$, we have $u \circ \phi(p_i) = q_i$.*

Proof. The first three properties follow from the conformal invariance of energy and the fact that linear fractional transformations are conformal diffeomorphisms of the disk. The last property uses that u is onto Γ and the linear fractional transformations are triply transitive on ∂D. □

We can now use the Courant-Lebesgue lemma to show that fixing the images of these three points and bounding the energy implies equicontinuity on the boundary:

Lemma 4.14. *For any constant K, define the family of maps $\mathcal{F} = \mathcal{F}_K$ to be all $\psi : D \to \mathbb{R}^3$ with*

(1) *$\psi \in C^0(\overline{D}) \cap W^{1,2}(D)$ has $E(\psi) \leq K/2$.*

(2) *$\psi : \partial D \to \Gamma$ is monotone and onto.*

(3) *For each $i = 1, 2, 3$, we have $\psi(p_i) = q_i$.*

Then $\mathcal{F}$ is equicontinuous on ∂D. Hence, by the Arzela-Ascoli theorem, $\mathcal{F}$ is compact in the topology of uniform convergence on ∂D.

Proof. Suppose that we are given $\epsilon > 0$; without loss of generality, we can take ϵ smaller than $\min |q_i - q_j|$.

Since Γ is a simple closed curve of finite length, it follows that there exists some $d > 0$ such that if $p, q \in \Gamma$ with $0 < |p - q| < d$, then

$$\Gamma \setminus \{p, q\} \text{ has exactly one component with diameter at most } \epsilon.$$

Choose some $\delta < 1$ such that $\sqrt{2\pi\epsilon_\delta} < d$ and such that given any $p \in \partial D$ at least two of the p_i are not in the ball of radius $\sqrt{\delta}$ about p.

Now, given any $p \in \partial D$, the Courant-Lebesgue lemma, i.e., Lemma 4.11, implies that there is some ρ between δ and $\sqrt{\delta}$ such that $d(C_\rho) < d$. The curve C_ρ divides ∂D into two components, A^1 and A^2, with the larger one (say, A^2) containing at least two of the base points p_i. Denote the corresponding arcs on Γ by $\mathcal{A}^1$ and $\mathcal{A}^2$. Since the image of the endpoints of C_ρ are connected by a curve of length less than d, one of $\mathcal{A}^1$ and $\mathcal{A}^2$ has diameter less than ϵ. Note that this component cannot contain two of the base points q_i, and we can thus conclude that $\mathcal{A}^1$ has diameter less than ϵ. Since $\mathcal{A}^1$ is the image of A^1, this completes the proof of equicontinuity. □

We are now prepared to solve the Plateau problem.

Proof of Theorem 4.1. We observe first that $E_\Gamma < \infty$. Namely, since Γ is piecewise C^1, we have a piecewise C^1 monotone, onto, map $w : \partial D \to \Gamma$. Working in polar coordinates (ρ, θ), we set

$$\tilde{w}(\rho, \theta) = \eta(\rho)\, w(\theta)\,,$$

where η is a smooth function with $\eta(1) = 1$ and $\eta(\rho) = 0$ for all $\rho < 1/2$. This gives a Lipschitz, and thus finite energy, map; therefore $E_\Gamma < \infty$.

By Proposition 4.5, there exists a minimizing sequence $\{u_j\}$ of harmonic maps that are in $C^0(\overline{D}) \cap W^{1,2}(D)$, so that $u_j : \partial D \to \Gamma$ is monotone and onto, and so that

$$\mathrm{E}(u_j) \to E_\Gamma\,.$$

Moreover, Lemma 4.13 allows us to take each $u_j \in \mathcal{F}$.

By Rellich's compactness, Lemma 4.6, there is a weakly convergent subsequence $u_k \to u \in W^{1,2}$ with

$$\mathrm{E}(u) \leq \liminf\{\mathrm{E}(u_k)\} = E_\Gamma\,.$$

The Courant-Lebesgue Lemma and its corollary, Lemma 4.14, imply that the boundary values of the u_j's form an equicontinuous family of functions on ∂D. Thus, Arzela-Ascoli gives a subsequence that converges uniformly on ∂D to a continuous function. Moreover, since each u_j is harmonic, so is $(u_j - u_k)$ and the maximum principle gives that

$$\sup_D |u_j - u_k| = \max_{\partial D} |u_j - u_k| .$$

Consequently, the uniform convergence on ∂D implies uniform convergence on $\bar{D}$. We conclude that

$$\mathrm{E}(u) = E_\Gamma$$

and u is harmonic. Moreover, the uniform convergence also implies that $u : \partial D \to \Gamma$ is monotone and onto. Finally, Lemma 4.4 implies that Area$(u) = A_\Gamma$ and that u is almost conformal. □

Recall that map u is said to be monotone on ∂D if the inverse image of a connected set is connected. For future reference, it is convenient to make the following definition:

Definition 4.15. Let Γ be a piecewise C^1 closed Jordan curve and let $u : \bar{D} \to \mathbb{R}^3$ be the $C^0(\overline{D}) \cap W^{1,2}(D)$ almost conformal harmonic map whose restriction to ∂D is a monotone and onto map to Γ. If u minimizes energy among all such maps, we will say that u is a *solution of the Plateau problem.*

It remains to discuss the regularity of u up to the boundary ∂D. General arguments for boundary regularity (applied in a wide variety of settings) were given by S. Hildebrandt in [**Hi**]. Since our primary concern is with the case of minimal surfaces in $\mathbb{R}^3$, we recall the following result of J. C. C. Nitsche [**Ni1**]:

Theorem 4.16 (Nitsche, [**Ni1**]). *If Γ is a regular Jordan curve of class $C^{k,\alpha}$ where $k \geq 1$ and $0 < \alpha < 1$, then a solution u of the Plateau problem is $C^{k,\alpha}$ on all of $\bar{D}$.*

The book [**Ni2**] contains a proof of Theorem 4.16 and an extensive discussion of boundary regularity.

3.1. Mappings from planar domains. The Plateau problem can be generalized to mappings from other planar domains with more than one boundary component. The simplest case is for maps from an annulus, where Meeks and Yau proved the following:

Theorem 4.17 (Meeks-Yau, [**MeY1**]). *Let Γ_1 and Γ_2 be simple closed curves. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the areas of the least area disks with boundary Γ_1 and Γ_2, respectively. If there is an annulus Σ with $\partial\Sigma = \Gamma_1 \cup \Gamma_2$ and*

$$\mathrm{Area}(\Sigma) < \mathcal{A}_1 + \mathcal{A}_2 ,$$

then there is a least area annulus with boundary $\Gamma_1 \cup \Gamma_2$.

We refer the reader to theorem 1 of [**MeY1**] for a proof of this and its generalization to other planar domains.

4. Branch Points

We will next show that the solution to the Plateau problem given in the previous section is, in fact, immersed. In this book, a *branch point* for a minimal surface

$$u : D \to \mathbb{R}^3$$

is a point where the differential of u vanishes; these are precisely the points where u fails to be an immersion. Thus, we wish to show that solutions to the Plateau problem do not have branch points; in contrast, there are minimal surfaces with branch points, but they cannot be energy minimizing maps from a disk.

Since u is automatically almost conformal, branch points of a minimal surface have the structure of branched conformal maps $z \to z^k$ for $k > 1$ in complex analysis (see, e.g., [**A**]) but they also have additional structure coming from the third coordinate function of the immersion.

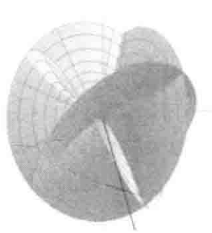

Figure 4.1. A minimal surface near a branch point. Figure courtesy of Michael Beeson.

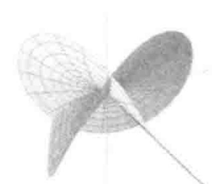

Figure 4.2. Half of a minimal surface near a branch point. Figure courtesy of Michael Beeson.

To see that the solutions to the Plateau problem are immersed, we first show that branch points are isolated (which is true for any minimal surface). Then, using the minimizing property, we show that interior branch points do not exist. It follows that the solution is an immersed surface.

If we let $z = (x + i\,y)$ be the complex coordinate on D, then we saw in the proof of Corollary 4.9 that the harmonicity of u implies that $v \equiv u_x - i\,u_y$

is a (vector-valued) holomorphic function. Furthermore, we have

$$|v|^2 = |u_x|^2 + |u_y|^2 , \tag{4.30}$$

and hence v vanishes precisely at the branch points of u. Since the zeros of a nonconstant holomorphic function on D must be isolated, this gives the following:

Corollary 4.18. *An almost conformal harmonic map $u : D \to \mathbb{R}^3$ has isolated branch points.*

As mentioned above, one can prove quite a bit more. R. Osserman showed that the image surface was immersed (namely, that there are no interior geometric branch points). R. Gulliver proved the definitive result: The solution has no interior branch points (so that not only is the image an immersed surface, but the parameterization does not introduce any branch points); cf. H. W. Alt. We will prove these results in Chapter 6.

5. Harmonic Maps

In the next two sections, M will be a closed Riemannian manifold isometrically embedded into some Euclidean space $\mathbb{R}^N$. Given a surface Σ and a map

$$u : \Sigma \to M , \tag{4.31}$$

the energy $\mathrm{E}(u)$ of the map u is defined to be

$$\mathrm{E}(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 , \tag{4.32}$$

where ∇ is the gradient on Σ. The map u is *harmonic* if it is a critical point for E with respect to compactly supported variations that map into M.

Arguing as in the proof of Lemma 4.12, it is easy to see that the energy of a map is conformally invariant in two dimensions:

Corollary 4.19. *If $\Phi : \Gamma \to \Sigma$ is a conformal diffeomorphism and $u : \Sigma \to M$ is harmonic, then $u \circ \Phi : \Gamma \to M$ is also harmonic and* $\mathrm{E}(u) = \mathrm{E}(u \circ \Phi)$.

5.1. The harmonic map equation.

We next derive the Euler-Lagrange equation for the energy functional; this is called the harmonic map equation.

Lemma 4.20. *A harmonic map $u : \Sigma \to M$ satisfies*

$$(\Delta u)^T = 0 , \tag{4.33}$$

where $(\Delta u)^T$ is the tangential projection to M. In particular,

$$\Delta_M u^i = g^{jk} A^i_{u(x)} (\partial_j u, \partial_k u) , \tag{4.34}$$

where u^i are the components of u, g_{jk} is the metric on Σ, and $A^i_{u(x)}$ is the i-th component of the second fundamental form of M at the point $u(x)$.

Proof. Let $w : \Sigma \times \mathbb{R} \to M$ be a one-parameter family of maps with $w(x,0) = u(x)$. Differentiating the energy gives

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathrm{E}(w(\cdot,t)) &= \int_\Sigma \langle \nabla w(\cdot,0), \nabla \partial_t w(\cdot,0)\rangle \\ &= -\int_\Sigma \langle \Delta u, \partial_t w(\cdot,0)\rangle \,, \end{aligned} \tag{4.35}$$

where the second equality used the divergence theorem. The only constraint on $\partial_t w(\cdot,0)$ is that it must be tangent to M since w always maps into M, so we get (4.33).

The derivation of (4.34) follows since Riemannian geometry implies that the normal part of Δu always can be expressed this way (this does not use the first equation). Namely, if N is a normal vector field along M, we have

$$\begin{aligned} \langle \Delta u, N \circ u\rangle &= \frac{1}{\sqrt{\det g}} \left\langle \partial_j \left(g^{jk} \sqrt{\det g}\, u_k\right), N\circ u\right\rangle \\ &= -g^{jk}\langle u_k, \partial_j \left(N\circ u\right)\rangle \,, \end{aligned} \tag{4.36}$$

where the second equality used metric compatibility and that u_k is tangent to M while N is normal to M. The chain rule implies that

$$\partial_j \left(N\circ u\right) = \nabla_{u_j} N \,, \tag{4.37}$$

so we can rewrite this as

$$\langle \Delta u, N\circ u\rangle = -g^{jk}\langle u_k, \nabla_{u_j} N\rangle = g^{jk}\langle \nabla_{u_j} u_k, N\rangle \,, \tag{4.38}$$

and this gives (4.34). □

Using this, we can derive a differential inequality for $|\nabla u|^2$ that plays the same role for harmonic maps that Simons' inequality does for minimal surfaces.

Lemma 4.21. *If $u : \Sigma \to M$ is harmonic, then there exists C depending only on M and a lower bound for* Ric_Σ *so that*

$$\Delta\, |\nabla u|^2 \geq -C\left(|\nabla u|^2 + |\nabla u|^4\right) . \tag{4.39}$$

Proof. Throughout the proof C will be a constant depending only on M and a lower bound for Ric_Σ and will be allowed to change from line to line.

We start with the Bochner formula (Proposition 1.47) which gives (applied to each component of u) that

$$\begin{aligned} \frac{1}{2}\,\Delta\,|\nabla u|^2 &= |\mathrm{Hess}_u|^2 + \mathrm{Ric}_\Sigma(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle \\ &\geq |\mathrm{Hess}_u|^2 - C\,|\nabla u|^2 - |\nabla u|\;|\nabla \Delta u|\,, \end{aligned} \tag{4.40}$$

where the inequality used that Σ is compact and thus has bounded Ricci curvature. Differentiating the harmonic map equation (4.34) gives that

$$\begin{aligned} |\nabla \Delta u| &\leq |\nabla (A\circ u)|\;|\nabla u|^2 + 2\,|A|\,|\nabla u|\;|\mathrm{Hess}_u| \\ &\leq C\;|\nabla u|^3 + C\,|\nabla u|\;|\mathrm{Hess}_u|\,, \end{aligned} \tag{4.41}$$

where the second inequality used that $|A|$ and $|\nabla A|$ are bounded for the closed manifold $M \subset \mathbb{R}^N$. Therefore, the absorbing inequality gives

$$\begin{aligned} |\nabla u|\;|\nabla \Delta u| &\leq C\;|\nabla u|^4 + C\,|\nabla u|^2\;|\mathrm{Hess}_u| \\ &\leq C'\;|\nabla u|^4 + |\mathrm{Hess}_u|^2\,. \end{aligned} \tag{4.42}$$

Finally, substituting this back into (4.40) and noting that the hessian terms cancel, gives the lemma. □

5.2. A priori estimates for harmonic maps. Using the differential inequality from Lemma 4.21 and arguing as in proof of Theorem 2.2, we get the small energy estimate of Sacks and Uhlenbeck, [**SaUh**]:

Theorem 4.22 (Sacks-Uhlenbeck, [**SaUh**]). *Given Σ and M, there exist $\epsilon, \rho > 0$ so that if $r_0 < \rho$, $u : \Sigma \to M$ is harmonic and*

$$\int_{B^\Sigma_{r_0}(y)} |\nabla u|^2 < \delta\,\epsilon\,, \tag{4.43}$$

then

$$|\nabla u|^2(y) \leq \frac{\delta}{r_0^2}\,. \tag{4.44}$$

Proof. Set $F = (r_0 - r)^2\,|\nabla u|^2$ on $B^\Sigma_{r_0}(y)$ where r is the distance to y and observe that F vanishes on $\partial B^\Sigma_{r_0}(y)$. Let $x_0 \in \Sigma$ be a point where F achieves its maximum and note that it is enough to show that $F(x_0) < \delta$. We will show that $F(x_0) \geq \delta$ leads to a contradiction for $\epsilon > 0$ sufficiently small.

Define $\sigma > 0$ by

$$\sigma^2\,|\nabla u|^2(x_0) = \frac{\delta}{4}\,. \tag{4.45}$$

In particular, since $F(x_0) \geq \delta$, we see that

$$2\,\sigma \leq r_0 - r(x_0)\,. \tag{4.46}$$

Consequently, by the triangle inequality, we have on $B^{\Sigma}_{\sigma}(x_0)$ that

$$\frac{1}{2} \leq \frac{r_0 - r}{r_0 - r(x_0)} \leq 2\,.$$

Since F achieves its maximum at x_0, we conclude that

$$\begin{aligned}(r_0 - r(x_0))^2 \sup_{B^{\Sigma}_{\sigma}(x_0)} |\nabla u|^2 &\leq 4 \sup_{B^{\Sigma}_{\sigma}(x_0)} F(x) = 4\,F(x_0) \\ &= 4\,(r_0 - r(x_0))^2\,|\nabla u|^2(x_0)\,. \end{aligned} \tag{4.47}$$

Dividing through by $(r_0 - r(x_0))^2$ and using the definition of σ gives

$$\sup_{B^{\Sigma}_{\sigma}(x_0)} |\nabla u|^2 \leq 4\,|\nabla u|^2(x_0) = \delta\,\sigma^{-2}\,. \tag{4.48}$$

After rescaling $B^{\Sigma}_{\sigma}(x_0)$ to unit size (and still calling it Σ!), we have

$$\sup_{B^{\Sigma}_{1}(x_0)} |\nabla u|^2 \leq 4\,|\nabla u|^2(x_0) = \delta \leq 1\,.$$

By Lemma 4.21 on $B^{\Sigma}_1(x_0)$,

$$\Delta|\nabla u|^2 \geq -2\,C\,|\nabla u|^2\,,$$

where C depends only on M and the lower bound for Ric_{Σ}. The desired contradiction now follows from the mean value inequality. Namely, a variation on Corollary 1.16 (see [**LiSc**] for the precise mean value inequality) gives

$$\frac{\delta}{4} = |\nabla u|^2(x_0) \leq C_1 \int_{B^{\Sigma}_1(x_0)} |\nabla u|^2 < C_1\,\delta\,\epsilon\,, \tag{4.49}$$

where C_1 depends only on M and the lower bound for the Ric_{Σ}. This is a contradiction provided that ϵ is chosen sufficiently small. □

Remark 4.23. *The previous estimate was proven for smooth harmonic maps. F. Hélein proved in* [**Hf2**] *that weakly harmonic maps are automatically smooth; cf.* [**Rt**], [**RtSt**].

Note that estimates on higher derivatives follow immediately from the L^{∞} estimate on $|\nabla u|$ and linear elliptic theory. Namely, the equation then implies that Δu is in L^{∞}, so linear elliptic theory (theorem 9.11 in [**GiTr**]) imply that u is in $W^{2,p}$ for any $p < \infty$. The derivative of the nonlinearity

$$A \circ u(\nabla u, \nabla u) \tag{4.50}$$

is bounded by (a constant times)

$$|\nabla u|^3 + |\nabla u|^2\,|\,\text{Hess}_u\,| \tag{4.51}$$

which is in L^p. It follows that Δu is in $W^{1,p}$ and thus u is in $W^{3,p}$ (this is called "boot-strapping"); see theorem 9.19 in [**GiTr**]. Repeating this shows

that u is in $W^{k,p}$ for every k. The Morrey-Sobolev embedding theorem (theorem 7.26 in [**GiTr**]) implies that u is smooth.

5.3. The Hopf differential. In this section, we will show that every finite energy harmonic map from $\mathbb{R}^2$ must also be conformal and, thus, gives a minimal surface in the target space M. Because of the conformal invariance of energy, it follows that the same is true for harmonic maps from $\mathbf{S}^2$.

The key is that the failure to be conformal is measured by a holomorphic differential called the Hopf differential and these must vanish on $\mathbf{S}^2$ or $\mathbb{R}^2$ (with a growth bound).

We will need the following simple lemma:

Lemma 4.24. *If ϕ is an L^1 holomorphic function on $\mathbb{R}^2$, then $\phi \equiv 0$.*

Proof. If f and g are the real and imaginary parts of ϕ, then

$$\begin{aligned} \Delta\,|\phi| &= \Delta\,\left(f^2+g^2\right)^{\frac{1}{2}} = \operatorname{div}\left(\frac{f\nabla f + g\nabla g}{(f^2+g^2)^{\frac{1}{2}}}\right) \\ &= \frac{|\nabla f|^2+|\nabla g|^2}{(f^2+g^2)^{\frac{1}{2}}} - \frac{|f\nabla f+g\nabla g|^2}{(f^2+g^2)^{\frac{3}{2}}}, \end{aligned} \tag{4.52}$$

where the second equality used that f and g are harmonic. The Cauchy Riemann equations give that ∇f and ∇g are perpendicular and $|\nabla f| = |\nabla g|$. Using this in the expression for $\Delta\,|\phi|$ gives

$$\Delta\,|\phi| = \frac{2\,|\nabla f|^2}{(f^2+g^2)^{\frac{1}{2}}} - \frac{|\nabla f|^2\,(f^2+g^2)}{(f^2+g^2)^{\frac{3}{2}}} = \frac{|\nabla f|^2}{(f^2+g^2)^{\frac{1}{2}}}\,. \tag{4.53}$$

It follows that $\Delta|\phi| \geq 0$, so the mean value inequality of Corollary 1.17 implies that either $\phi \equiv 0$, or the $L^1(B_R)$ norm grows at least quadratically in R. The lemma follows. □

The next lemma shows that finite energy harmonic maps from $\mathbb{R}^2$ are conformal; by the conformal invariance of energy in dimension two, it follows that harmonic maps from $\mathbf{S}^2$ are conformal. This is due to Sacks and Uhlenbeck (corollary 1.7 in [**SaUh**]).

Lemma 4.25 (Sacks-Uhlenbeck, [**SaUh**]). *If $u : \mathbb{R}^2 \to M$ is harmonic and $\mathrm{E}(u) < \infty$, then u is almost conformal and minimal.*

Proof. Consider the complex-valued function

$$\phi(x,y) = (|u_x|^2 - |u_y|^2) - 2\,i\,\langle u_x, u_y\rangle\,. \tag{4.54}$$

We will see that ϕ is holomorphic. We compute

$$(|u_x|^2 - |u_y|^2)_x = 2\langle u_x, u_{xx}\rangle - 2\langle u_y, u_{xy}\rangle\,, \tag{4.55}$$

$$(|u_x|^2 - |u_y|^2)_y = 2\langle u_x, u_{xy}\rangle - 2\langle u_y, u_{yy}\rangle\,, \tag{4.56}$$

$$(\langle u_x, u_y\rangle)_x = \langle u_{xx}, u_y\rangle + \langle u_x, u_{yx}\rangle\,, \tag{4.57}$$

$$(\langle u_x, u_y\rangle)_y = \langle u_{xy}, u_y\rangle + \langle u_x, u_{yy}\rangle\,. \tag{4.58}$$

Putting these into the Cauchy-Riemann equations gives

$$\begin{aligned}(|u_x|^2 - |u_y|^2)_x - (-2\langle u_x, u_y\rangle)_y &= 2\,(\langle u_x, u_{xx}\rangle - \langle u_y, u_{xy}\rangle \\ &\quad + \langle u_{xy}, u_y\rangle + \langle u_x, u_{yy}\rangle) \\ &= 2\langle \Delta u, u_x\rangle = 0\,,\end{aligned} \tag{4.59}$$

where the last equality used that u_x is tangential to M and Δu is perpendicular to M by the harmonic map equation. Similarly, we get that

$$(|u_x|^2 - |u_y|^2)_y + (-2\langle u_x, u_y\rangle)_x = -2\langle \Delta u, u_y\rangle = 0\,, \tag{4.60}$$

so we conclude that ϕ is holomorphic. Since $\mathrm{E}(u) < \infty$, the function ϕ is in $L^1(\mathbb{R}^2)$. Finally, by Lemma 4.24, $\phi \equiv 0$ and we conclude that u is almost conformal. □

5.4. Removable singularities. We will use the following removable singularity result for harmonic maps from surfaces:

Theorem 4.26 (Sacks-Uhlenbeck, [**SaUh**]). *If $u : B_1 \setminus \{0\} \to M$, where $B_1 \subset \mathbb{R}^2$, is a smooth harmonic map with finite energy, then u extends smoothly as a harmonic map from B_1 to M.*

We refer to theorem 3.6 in [**ScYa2**] for the proof.

6. Existence of Minimal Spheres in a Homotopy Class

In this section, we will prove that every closed manifold M with nontrivial $\pi_2(M)$ must contain a minimal sphere. This was proven by Sacks and Uhlenbeck in [**SaUh**] and we will follow their approach of finding minimizers for perturbed functionals and taking the limit as the perturbation goes to zero. The argument of Sacks and Uhlenbeck gives existence when $\pi_k(M) \neq 0$ for some $k \geq 2$:

Theorem 4.27 (Sacks-Uhlenbeck, [**SaUh**]). *If M is a closed manifold and $\pi_k(M) \neq 0$ for some $k \geq 2$, then there is a nontrivial harmonic map $u : \mathbf{S}^2 \to M$. The map u is almost conformal and its image is minimal.*

The rest of this section will be devoted to proving Theorem 4.27 in the case $\pi_2(M) \neq 0$. We will not do the case $\pi_k(M) \neq 0$ for $k > 2$ here since this

result (and in fact much more) follows from the min-max existence results in the next chapter.

When $\pi_2(M) \neq 0$, the natural approach would be to minimize energy in the homotopy class. However, the space of $W^{1,2}$ maps with bounded energy is not compact and a minimizing sequence need not converge (even after passing to a subsequence). We already saw this lack of compactness in the Plateau problem for disks, where it came from the noncompactness of the group of conformal automorphisms of the disk. The same is true for maps from $\mathbf{S}^2$ where the group of Möbius transformations is noncompact.

Sacks and Uhlenbeck overcame this difficulty by first minimizing a perturbed functional E_α that is not invariant under the conformal group and where a minimizing sequence must converge to a minimizer in the same homotopy class. The main point will be to get uniform control on the minimizers of the perturbed functionals so that we can take the limit.

6.1. The perturbed functionals. We will next define the perturbed functionals of Sacks and Uhlenbeck and deduce the Euler-Lagrange equation for these functionals. Namely, given $\alpha \geq 1$ and a map $u : \mathbf{S}^2 \to M$, define a perturbed energy function

$$E_\alpha(u) = \int_{\mathbf{S}^2} \left(1 + |\nabla u|^2\right)^\alpha . \tag{4.61}$$

Clearly, when $\alpha = 1$, we have $E_1(u) = 4\pi + 2\,\mathrm{E}(u)$ so the critical points of E_1 are precisely the harmonic maps.

Following Lemma 4.20, we compute the Euler-Lagrange equation for E_α.

Lemma 4.28. *If the map $u : \Sigma \to M$ is a critical point for E_α, then*

$$\left(\operatorname{div}\left((1 + |\nabla u|^2)^{\alpha-1}\, \nabla u\right)\right)^T = 0\,, \tag{4.62}$$

where $(\cdot)^T$ is the tangential projection to M. In particular,

$$\Delta u^i + (\alpha - 1)\, \frac{\langle \nabla\, |\nabla u|^2\,, \nabla u^i\rangle}{1 + |\nabla u|^2} = g^{jk} A^i_{u(x)}\left(\partial_j u, \partial_k u\right)\,, \tag{4.63}$$

where u^i are the components of u, g_{jk} is the metric on Σ, and $A^i_{u(x)}$ is the i-th component of the second fundamental form of M at the point $u(x)$.

Proof. Let $w : \Sigma \times \mathbb{R} \to M$ be a one-parameter family of maps with $w(x, 0) = u(x)$. Differentiating E_α gives

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} E_\alpha(w(\cdot, t)) &= \alpha \int_\Sigma (1 + |\nabla w|^2(\cdot, 0))^{\alpha-1} \langle \nabla w(\cdot, 0), \nabla \partial_t w(\cdot, 0)\rangle \\ &= -\alpha \int_\Sigma \langle \operatorname{div}\left((1 + |\nabla u|^2)^{\alpha-1}\, \nabla u\right), \partial_t w(\cdot, 0)\rangle\,, \end{aligned} \tag{4.64}$$

where the second equality used the divergence theorem. The only constraint on the variation $\partial_t w(\cdot, 0)$ is that it must be tangent to M since w always maps into M, so we get (4.62).

To get (4.63), let N is a normal vector field along M so that

$$
\begin{aligned}
&\langle \operatorname{div}\left((1+|\nabla u|^2)^{\alpha-1}\,\nabla u\right), N\circ u\rangle \\
&= \frac{1}{\sqrt{\det g}}\,\langle \partial_j \left((1+|\nabla u|^2)^{\alpha-1} g^{jk}\,\sqrt{\det g}\, u_k\right), N\circ u\rangle \\
&= -g^{jk}\,(1+|\nabla u|^2)^{\alpha-1}\,\langle u_k, \partial_j\,(N\circ u)\rangle\,, \qquad (4.65)
\end{aligned}
$$

where the second equality used metric compatibility and that u_k is tangent to M while N is normal to M. The chain rule implies that

$$\partial_j\,(N\circ u) = \nabla_{u_j} N\,, \qquad (4.66)$$

so we can rewrite this as

$$
\begin{aligned}
\langle \operatorname{div}\left((1+|\nabla u|^2)^{\alpha-1}\,\nabla u\right), N\circ u\rangle &= -g^{jk}\,(1+|\nabla u|^2)^{\alpha-1}\,\langle u_k, \nabla_{u_j} N\rangle \\
&= g^{jk}\,(1+|\nabla u|^2)^{\alpha-1}\,\langle \nabla_{u_j} u_k, N\rangle\,. \qquad (4.67)
\end{aligned}
$$

Since $\operatorname{div}\left((1+|\nabla u|^2)^{\alpha-1}\,\nabla u\right)$ is equal to

$$(1+|\nabla u|^2)^{\alpha-1}\,\Delta u + (\alpha-1)\,(1+|\nabla u|^2)^{\alpha-2}\,\langle \nabla\,|\nabla u|^2\,, \nabla u\rangle\,, \qquad (4.68)$$

this gives (4.63). □

The key to obtaining a limiting harmonic map from a sequence of critical points for the perturbed functionals is to get estimates that are independent of α. The next lemma gives such an estimate:

Lemma 4.29. *Given M and $p \in (0,\infty)$, there exist $\alpha_0 > 1$ and C so that if $u: \mathbf{S}^2 \to M$ is critical for E_α with $\alpha \leq \alpha_0$, then*

$$||u||_{W^{2,p}} \leq C\left[1 + \left(\sup|\nabla u|^2\right)^{\frac{p-1}{p}}\left(\int_{\mathbf{S}^2}|\nabla u|^2\right)^{\frac{1}{p}}\right]. \qquad (4.69)$$

Proof. The Euler-Lagrange equation gives that

$$\left|\Delta u^i\right| \leq 2\,(\alpha-1)\,|\nabla|\nabla u|| + C_M\,|\nabla u|^2 \leq 2\,(\alpha-1)\,|\nabla^2 u| + C_M\,|\nabla u|^2\,,$$

where the second inequality used the (vector) Cauchy-Schwarz inequality[3] and C_M comes from the second fundamental form of M and does not depend on u.

[3]The inequality $|\nabla|\nabla u|| \leq |\nabla^2 u|$ is sometimes called the Kato inequality.

Given $p \in (1, \infty)$, linear elliptic theory (theorem 9.11 in **[GiTr]**) gives a uniform constant $\bar{C}$ so that for any unit ball $B_1 \subset \mathbf{S}^2$ we have

$$\begin{aligned} ||u^i||_{W^{2,p}(B_{1/2})} &\leq \bar{C}\, \left(||u||_{L^p(B_1)} + ||\Delta u||_{L^p(B_1)} \right) \\ &\leq \bar{C} \left(4\pi \,\mathrm{diam}(M) + C_M \left(\sup |\nabla u|^2\right)^{\frac{p-1}{p}} \left(\int_{B_1} |\nabla u|^2 \right)^{\frac{1}{p}} \right. \\ &\qquad \left. + 2\,(\alpha - 1)\, ||\nabla^2 u||_{L^p(B_1)} \right) . \end{aligned} \tag{4.70}$$

Summing this over i gives

$$\begin{aligned} ||u||_{W^{2,p}(B_{1/2})} &\leq C_1\, N + C_1\, N\, \left(\sup |\nabla u|^2\right)^{\frac{p-1}{p}} \left(\int_{\mathbf{S}^2} |\nabla u|^2 \right)^{\frac{1}{p}} \\ &\qquad + C_1\, N\, (\alpha - 1)\, ||\nabla^2 u||_{L^p(B_1)} , \end{aligned} \tag{4.71}$$

where N is the dimension of the Euclidean space that M sits in and C_1 depends only on p and M.

Using that $\mathbf{S}^2$ has the $\mathbb{R}^2$ volume doubling property, we can cover $\mathbf{S}^2$ by a collection of balls $B_{1/2}(y_i)$ of radius $1/2$ so that

$$\sum_i \chi_{B_1(y_i)} \leq 9 ,$$

where χ_E is the characteristic function of a set E (so $\chi_E(x) = 1$ if $x \in E$ and zero otherwise). Summing (4.71) over these balls gives

$$\begin{aligned} ||u||_{W^{2,p}(\mathbf{S}^2)} &\leq 9\, C_1\, N + 9\, C_1\, N\, \left(\sup |\nabla u|^2\right)^{\frac{p-1}{p}} \left(\int_{\mathbf{S}^2} |\nabla u|^2 \right)^{\frac{1}{p}} \\ &\qquad + 9\, C_1\, N\, (\alpha - 1)\, ||\nabla^2 u||_{L^p(\mathbf{S}^2)} . \end{aligned} \tag{4.72}$$

We choose $\alpha_0 > 1$ so that $9\, C_1\, N\, (\alpha_0 - 1) = 1/2$. Thus, we can absorb the $||\nabla^2 u||_{L^p}$ term on the right side of (4.72) to get

$$||u||_{W^{2,p}} \leq 18\, C_1\, N \left[1 + \left(\sup |\nabla u|^2\right)^{\frac{p-1}{p}} \left(\int_{\mathbf{S}^2} |\nabla u|^2 \right)^{\frac{1}{p}} \right] . \tag{4.73}$$

This gives the uniform $W^{2,p}$ estimate for u □

6.2. Minimizers for the perturbed functionals. We will next prove the existence of minimizers for the perturbed functionals in a homotopy class.

We start with a simple observation:

Lemma 4.30. *Given M, there exists $r_M > 0$ so that if u and v are continuous maps from $\mathbf{S}^2$ to M and*

$$\sup_{x \in \mathbf{S}^2} \mathrm{dist}_M(u(x), v(x)) \leq r_M ,$$

then u and v are homotopic.

Proof. Since M is compact and smooth, there exists $r_M > 0$ so that the exponential map is a diffeomorphism on every closed intrinsic ball of radius r_M. Define $\Gamma \subset M \times M$ by

$$\Gamma = \{(x, y) \in M \times M \mid \text{dist}_M(x, y) \leq r_M\},$$

and define a map

$$H : \Gamma \to C^1([0, 1], M)$$

by letting $H(x, y) : [0, 1] \to M$ be the linear map from x to y.

Setting $w(x, t) = H(u(x), v(x))(t)$ gives the desired homotopy. □

Proposition 4.31. *If $w : \mathbf{S}^2 \to M$ is Lipschitz and $\alpha > 1$, then there is a smooth map $u : \mathbf{S}^2 \to M$ that minimizes E_α in the homotopy class of w and satisfies* (4.63).

Proof. The Morrey-Sobolev embedding theorem (theorem 7.26 in [**GiTr**]) yields that any $W^{1,2\alpha}$ map is continuous and, in fact, is Hölder continuous with exponent

$$\frac{\alpha - 1}{\alpha}.$$

Thus, the homotopy class is well defined. Let $I = I(\alpha, w)$ be the infimum of E_α over all $W^{1,2\alpha}$ maps from $\mathbf{S}^2$ to M that are homotopic to w.

Let u_j be a sequence of $W^{1,2\alpha}$ maps from $\mathbf{S}^2$ to M that are homotopic to w and have

$$E_\alpha(u_j) \leq I + \frac{1}{j}. \tag{4.74}$$

These maps are equicontinuous since they are uniformly $C^{\frac{\alpha-1}{\alpha}}$, so Arzela-Ascoli gives a subsequence that converges uniformly to a limiting map u. Thus, by Lemma 4.30, u is homotopic to w. Using weak compactness of $W^{1,2\alpha}$ and lower semi-continuity of E_α, we see that u is also in $W^{1,2\alpha}$ and

$$E_\alpha(u) \leq \lim E_\alpha(u_j) = I. \tag{4.75}$$

It follows that $E_\alpha(u) = I$, as claimed.

Since u minimizes E_α relative to any variation (the variation is a homotopy and thus does not change the homotopy class), Lemma 4.28 implies that u is a $W^{1,2\alpha}$ weak solution to (4.63). Once we have this, linear elliptic theory takes over and gives that u is smooth (cf. [**Mo2**]). This would be completely standard if u was a scalar-valued function (since the equation could be written as a perturbation of the Laplacian), but requires some care here since the system of equations does not decouple when u is vector-valued and thus standard scalar linear theory does not apply. We will address this step differently than Sacks and Uhlenbeck. Namely, this can be proven by modifying the argument in Lemma 4.29 to get that u is in $W^{2,\alpha}$ and then

bootstrapping to get higher regularity. The main difference from Lemma 4.29 is that we do not bring in $\sup |\nabla u|$ in the second inequality in (4.70), but instead use that $|\nabla u|^2 \in L^\alpha$. $\square$

6.3. Limiting harmonic maps. The next corollary gives the existence of harmonic maps when there is a sequence of approximate harmonic maps with a uniform gradient bound.

Corollary 4.32 (Sacks-Uhlenbeck, [**SaUh**]). *If u_i are critical for E_{α_i} where $\alpha_i \to 1$ and*

$$\liminf_{i\to\infty} \left(\sup_{\mathbf{S}^2} |\nabla u_i| \right) < \infty \,, \tag{4.76}$$

then there is a subsequence that converges in $C^{1,1/2}$ to a smooth harmonic map $u : \mathbf{S}^2 \to M$. If the u_i's are nontrivial in $\pi_2(M)$, then so is u.

Proof. After passing to a subsequence, we can assume that $|\nabla u_i| < C_1 < \infty$. Taking $p > 3$, Lemma 4.29 then gives C_2 so that

$$||u_i||_{W^{2,p}} < C_2 \,,$$

and the Morrey-Sobolev embedding theorem gives a subsequence (still denoted by u_i) that converges in $C^{1,1/2}$ to a limiting function u.

By Lemma 4.28, each u_i solves the divergence form equation

$$\operatorname{div}\left((1 + |\nabla u_i|^2)^{\alpha_i - 1} \nabla u_i \right) = (1 + |\nabla u_i|^2)^{\alpha_i - 1} A \circ u_i(\nabla u_i, \nabla u_i) \,. \tag{4.77}$$

Since $u_i \to u$ in $C^{1,1/2}$ and $\alpha_i \to 1$, the right side of (4.77) converges uniformly to $A \circ u(\nabla u, \nabla u)$ and the vector field $(1+|\nabla u_i|^2)^{\alpha_i - 1} \nabla u_i$ converges uniformly to ∇u. It follows that u is a $C^{1,1/2}$ weak solution of the harmonic map equation

$$\operatorname{div}(\nabla u) = A \circ u(\nabla u, \nabla u) \,.$$

As we saw before, linear elliptic theory (theorems 9.11 and 9.19 in [**GiTr**]) then implies that u is a smooth solution.

Finally, since the u_i's converge to u uniformly, Lemma 4.30 implies that for i sufficiently large the u_i's are all homotopic to u. $\square$

6.4. Renormalization. The next lemma will give a compactness theorem for sequences of approximate harmonic maps with uniformly bounded energy. Following Sacks and Uhlenbeck, compactness will only hold after composing with a sequence of dilations and there may be energy lost in the limit.

Lemma 4.33 (Sacks-Uhlenbeck, [**SaUh**]). *Suppose that u_i are critical for E_{α_i}, $i \to \infty$, where $\alpha_i \to 1$, and the u_i's have bounded energy $\int |\nabla u_i|^2 \leq C_E$.*

If $\sup_{\mathbf{S}^2} |\nabla u_i| \to \infty$*, then there is a subsequence (still denoted by* u_i*), a sequence of rotations of* $\mathbf{S}^2$*, and a sequence of scale factors*

$$\lambda_i = \sup_{\mathbf{S}^2} |\nabla u_i|^2 \to \infty \tag{4.78}$$

so that $(\mathbf{S}^2, \lambda_i\, g) \to (\mathbb{R}^2, \delta_{ij})$ *and the* u_i*'s converge in* $C^{1,1/2}$ *to a smooth harmonic map* $u : \mathbb{R}^2 \to M$ *with* $|\nabla u|(0) = 1$ *and* $\int_{\mathbb{R}^2} |\nabla u|^2 \leq C_E$.

Before proving the lemma, we collect some simple facts about the conformal metric

$$\bar{g} = \lambda\, g$$

where g is the standard (constant curvature one) metric on $\mathbf{S}^2$ and $\lambda > 0$ is a constant. Since the dimension is two, the volume element changes by

$$dv_{\bar{g}} = \lambda\, dv_g\,. \tag{4.79}$$

If $v : \mathbf{S}^2 \to \mathbb{R}$ is a function, then

$$\nabla_{\bar{g}} v = \frac{1}{\lambda}\, \nabla_g v\,, \tag{4.80}$$

$$\operatorname{div}_{\bar{g}} = \operatorname{div}_g\,, \tag{4.81}$$

$$\Delta_{\bar{g}} v = \frac{1}{\lambda}\, \Delta_g v\,. \tag{4.82}$$

Using these transformation rules and the change in the metric gives

$$|\nabla_{\bar{g}} v|^2_{\bar{g}} = \frac{1}{\lambda}\, |\nabla_g v|^2_g\,, \tag{4.83}$$

$$|\operatorname{Hess}_{\bar{g}} v|^2_{\bar{g}} = \frac{1}{\lambda^2}\, |\operatorname{Hess}_g v|^2_g\,. \tag{4.84}$$

Given $q \geq 1$, combining (4.79) and (4.84) shows that

$$\int |\operatorname{Hess}_{\bar{g}} v|^q_{\bar{g}}\, dv_{\bar{g}} = \lambda^{1-q} \int |\operatorname{Hess}_g v|^q_g\, dv_g\,. \tag{4.85}$$

We are now prepared to prove Lemma 4.33.

Proof of Lemma 4.33. Define conformal metrics

$$g_i = \lambda_i\, g\,,$$

where g is the standard metric on $\mathbf{S}^2$ and λ_i is defined by (4.78). After rotating $\mathbf{S}^2$, we can assume that the supremum in (4.78) is achieved at the same point $z \in \mathbf{S}^2$ for all i. Since $\lambda_i \to \infty$, this sequence of manifolds converges smoothly to flat Euclidean space; we can assume that $z \to 0$.

Using the transformation rule (4.83), the map u_i satisfies

$$\sup_{\mathbf{S}^2} |\nabla_{g_i} u|_{g_i} = |\nabla_{g_i} u|_{g_i}(z) = 1\,. \tag{4.86}$$

Given $p > 3$ and i large enough, Lemma 4.29 gives C_p so that

$$\int |\mathrm{Hess}_g\, u_i|_g^p \, dv_g \le C_p \left(1 + \sup |\nabla_g u_i|_g^{2p-2} \int |\nabla_g u_i|_g^2 \, dv_g\right)$$
$$\le C_p(1 + \lambda_i^{p-1}\, C_E)\,,$$

where the last inequality used the definition of λ_i and the uniform energy bound for the u_i's. Combining this with the transformation rule (4.85) for the norm of the Hessian, we get

$$\begin{aligned} \int |\mathrm{Hess}_{g_i}\, u_i|_{g_i}^p \, dv_{g_i} &= \lambda_i^{1-p} \int |\mathrm{Hess}_g\, u_i|_g^p \, dv_g \quad \le C_p\, \lambda_i^{1-p}\,(1 + \lambda_i^{p-1}\, C_E) \\ &\le 2\, C_p\, C_E\,, \end{aligned} \tag{4.87}$$

where the last inequality assumes that $\lambda_i \ge 1$ and $C_E \ge 1$ (which can be assumed).

Now that we have a uniform bound on the L^p norm of the Hessian for $p > 3$, we can proceed as in the proof of Lemma 4.32. Namely, the Morrey-Sobolev embedding theorem gives a subsequence (still denoted by u_i) that converges in $C^{1,1/2}$ (on compact subsets) to a limiting map $u : \mathbb{R}^2 \to M$. Furthermore, (4.86) and the uniform convergence of the gradient imply that $|\nabla u|(0) = 1$ and the lower semi-continuity of energy implies that $\int_{\mathbb{R}^2} |\nabla u|^2 \le C_E$.

In the remainder of this proof, we will use $\mathcal{L}_g$ to denote the harmonic map operator

$$\mathcal{L}_g v^\ell = \Delta_g v^\ell - g^{jk}\, A^\ell \circ v(\partial_j v, \partial_k v)\,,$$

so that v is harmonic precisely when $\mathcal{L}_g v = 0$. Similarly, $\mathcal{L}_{g_i}$ denotes the harmonic map operator with respect to the metric g_i. Using the transformation rules (4.80) and (4.82), we see that

$$\mathcal{L}_g = \lambda_i\, \mathcal{L}_{g_i}\,.$$

(This is consistent with the conformal invariance of harmonic maps in dimension two.)

By Lemma 4.28, each u_i satisfies

$$\mathcal{L}_g u_i = (1 - \alpha_i)\, \frac{\langle \nabla_g |\nabla_g u_i|_g^2, \nabla_g u_i \rangle_g}{1 + |\nabla_g u_i|_g^2}\,.$$

Using the transformation rules (4.80) and (4.83) gives

$$\lambda_i\, \mathcal{L}_{g_i} u_i = (1 - \alpha_i)\, \frac{\lambda_i^2\, \langle \nabla_{g_i} |\nabla_{g_i} u_i|_{g_i}^2, \nabla_{g_i} u_i \rangle_{g_i}}{1 + \lambda_i\, |\nabla_{g_i} u_i|_{g_i}^2}\,,$$

so we conclude that

$$
\begin{aligned}
|\mathcal{L}_{g_i} u_i| &\leq (1-\alpha_i)\, \frac{2\,\lambda_i\, |\nabla_{g_i} u_i|_{g_i}^2\, |\operatorname{Hess}_{g_i} u_i|_{g_i}}{1+\lambda_i\, |\nabla_{g_i} u_i|_{g_i}^2} \\
&\leq 2\,(1-\alpha_i)\, |\operatorname{Hess}_{g_i} u_i|_{g_i}\,. \qquad (4.88)
\end{aligned}
$$

Since $\alpha_i \to 1$, it follows from (4.87) that the right side of (4.88) goes to zero in L^p. Thus, we have that

$$
\mathcal{L}_{g_i} u_i \to 0 \text{ in } L^p\,.
$$

On the other hand, it is easy to see that $\mathcal{L}_{g_i} u_i \to \mathcal{L}_{\mathbb{R}^2} u$ weakly on each compact set $\Omega \subset \mathbb{R}^2$. Namely, the $C^{1,1/2}$ convergence of u_i to u implies that the nonlinearities converge uniformly and that the (divergence form operators) $\Delta_{g_i} u_i$ converge weakly to $\Delta_{\mathbb{R}^2} u$. Therefore, we conclude that u is a $C^{1,1/2}$ weak solution of $\mathcal{L}_{\mathbb{R}^2} u = 0$. As we saw before, linear elliptic theory (theorems 9.11 and 9.19 in [**GiTr**]) then implies that u is a smooth solution. □

We can now prove the Sacks-Uhlenbeck existence theorem:

Proof of Theorem 4.27 when $\pi_2(M) \neq 0$. Since $\pi_2(M) \neq 0$, we can choose a Lipschitz map $w : \mathbf{S}^2 \to M$ that is homotopically nontrivial. In particular, for every $\alpha \in [1,2]$ we have

$$
E_\alpha(w) = \int_{\mathbf{S}^2} (1+|\nabla w|^2)^\alpha \leq 4\pi\,(1+\sup|\nabla w|^2)^2\,. \qquad (4.89)
$$

For each $j > 0$, Proposition 4.31 gives a smooth map $u_j : \mathbf{S}^2 \to M$ that minimizes $E_{\frac{1+j}{j}}$ in the homotopy class of w and satisfies (4.63). In particular, the minimizing property and (4.89) give

$$
\int_{\mathbf{S}^2} |\nabla u_j|^2 \leq \int_{\mathbf{S}^2} (1+|\nabla u_j|^2)^{\frac{1+j}{j}} \leq E_{\frac{1+j}{j}}(w) \leq 4\pi\,(1+\sup|\nabla w|^2)^2\,. \qquad (4.90)
$$

We will divide into two cases, depending on whether or not the u_j's are blowing up.

Case 1: Suppose first that

$$
\liminf_{j\to\infty} \left(\sup_{\mathbf{S}^2} |\nabla u_j| \right) < \infty\,.
$$

In this case, Corollary 4.32 gives a subsequence that converges in $C^{1,1/2}$ to a smooth harmonic map $u : \mathbf{S}^2 \to M$ that is homotopic to w. In particular, u is nontrivial since the homotopy class is nontrivial.

Case 2: Suppose now that

$$
\liminf_{j\to\infty} \left(\sup_{\mathbf{S}^2} |\nabla u_j| \right) = \infty\,.
$$

In this case, Lemma 4.33 gives a subsequence (still denoted by u_j), a sequence of rotations of $\mathbf{S}^2$, and a sequence of scale factors

$$\lambda_j = \sup_{\mathbf{S}^2} |\nabla u_j|^2 \to \infty$$

so that $(\mathbf{S}^2, \lambda_j\, g) \to (\mathbb{R}^2, \delta_{ij})$ and the u_j's converge in $C^{1,1/2}$ to a smooth harmonic map $u : \mathbb{R}^2 \to M$ with $|\nabla u|(0) = 1$ and

$$\int_{\mathbb{R}^2} |\nabla u|^2 \leq 4\pi\, (1 + \sup |\nabla w|^2)^2 \,. \tag{4.91}$$

The energy bound comes from (4.90) and the conformal invariance of energy. Using stereographic projection and conformal invariance, u gives a nontrivial harmonic map from $\mathbf{S}^2 \setminus \{p\} \to M$ with finite energy where p is the point at infinity. By the removable singularity theorem, Theorem 4.26, u extends to a harmonic map on all of $\mathbf{S}^2$. □

As mentioned earlier, their argument actually works when $\pi_k(M) \neq 0$ for some $k \geq 2$. In this case, one would use a variational argument to get higher index critical points for the perturbed functionals (which satisfy the Palais-Smale Condition (C)) and then apply the rest of the argument without change. We will prove a stronger existence result in the next chapter.

Chapter 5

Min-max Constructions

In this chapter, we use a very general argument, whose basic idea goes back to H.A. Schwarz in the 1870s in his work on harmonic functions and G.D. Birkhoff in the 1910s in his work on closed geodesics, to find minimal spheres on any sphere. The treatment that we give here will follow the papers [**CM27**] and [**CM28**] where some of the existence results were new.

The idea of both Schwarz and Birkhoff was to use a min-max argument to show existence of critical points for variational problems. In this chapter, this idea allows us, in particular, to produce minimal surfaces that are not stable. In the min-max construction of minimal surfaces, one sweeps out the manifold by surfaces keeping track of the areas of the slices of the sweepout. One then tries to extract a convergent sequence of maximal slices for which the area of the maximal slices converge to the infimum of the maximal slices of all sweepouts. The idea is that, if this is done right, then such a sequence of maximal slices should converge in a weak topology to a nontrivial minimal surface. Even though such a construction is often referred to as a min-max construction it should in reality be called an inf-max construction as one takes the infimum over the maximum of the areas of all sweepouts.

1. Sweepouts by Curves

Given a Riemannian metric on the 2-sphere, sweep the 2-sphere out by a continuous one-parameter family of closed curves starting and ending at point curves. Pull the sweepout tight by, in a continuous way, pulling each curve as tight as possible yet preserving the sweepout.

Finding closed geodesics on the 2-sphere by using sweepouts goes back to Birkhoff in 1917; see [**B1**], [**B2**] *and section 2 in* [**Cr**] *for more about* Birkhoff's ideas. The argument works equally well on any closed manifold, but only produces nontrivial closed geodesics when the width, which is defined in (5.23) below, is positive. For instance, when M is topologically a 2-sphere, the width is, loosely speaking, up to a constant the square of the length of the shortest closed curve needed to "pull over" M. Birkhoff's argument gives that 2π times the width is realized as the length squared of a closed geodesic.

Throughout this chapter, M will be a closed Riemannian manifold isometrically embedded into some Euclidean space $\mathbb{R}^N$. It will be convenient to scale $\mathbb{R}^N$, and thus M, by a constant so that it satisfies the following three properties:

- (M1) $\sup_M |A| \leq 1/16$, where $|A|^2$ is the norm squared of the second fundamental form of M;
- (M2) the injectivity radius of M is at least 8π and the curvature is at most $1/64$, so that every geodesic ball of radius at most 4π in M is strictly geodesically convex;
- (M3) if $x, y \in M$ with $|x - y| \leq 1$, then $\text{dist}_M(x, y) \leq 2|x - y|$.

The last property is a chord-arc relating intrinsic and extrinsic distances.

2. Birkhoff's Curve Shortening Process

Let $W^{1,2}$ be the space of $W^{1,2}$ maps from $\mathbf{S}^1$ to M, where we identify $\mathbf{S}^1$ with $\mathbb{R}/2\pi\mathbf{Z}$. We will use the distance and topology on $W^{1,2}$ given by the $W^{1,2}$ (Sobolev) norm. Recall that the square of the $W^{1,2}$ norm of a map $f : \mathbf{S}^1 \to \mathbb{R}^N$ is

$$\int_{\mathbf{S}^1} \left(|f|^2 + |f'|^2 \right) .$$

We will use two basic facts about the space of curves in the $W^{1,2}$ norm.

The first basic fact is that the $W^{1,2}$ norm implies Hölder regularity. To see this, suppose that $x, y \in \mathbf{S}^1$ and use the fundamental theorem of calculus and Cauchy-Schwarz to get

$$\begin{aligned} |f(x) - f(y)|^2 &\leq \left(\int_x^y |f'| \right)^2 \leq \left(\int_x^y 1 \right) \left(\int_x^y |f'|^2 \right) \\ &\leq |x - y| \int_{\mathbf{S}^1} |f'|^2 . \end{aligned} \tag{5.1}$$

Thus, we see that f is in $C^{\frac{1}{2}}$. Thus, if f, g are $W^{1,2}$ close, then applying this to $(f - g)$ shows that they are C^0 close.

The second basic estimate that we will use repeatedly is the Wirtinger inequality. This is just the usual Dirichlet Poincaré inequality which bounds the L^2 norm in terms of the L^2 norm of the derivative; i.e.,

$$\int_0^{\pi L} f^2 dt \leq L^2 \int_0^{\pi L} (f')^2 dt \,,$$

provided $f(0) = f(\pi L) = 0$. Note that equality is achieved in Wirtinger's inequality when $f(t) = a \sin(t/L)$ for a constant a.

2.1. The space of curves and the curve shortening process. Fix a large positive integer L and let Λ denote the space of piecewise linear maps from $\mathbf{S}^1$ to M with exactly L breaks (possibly with unnecessary breaks) such that the length of each geodesic segment is at most 2π, parametrized by a (constant) multiple of arclength, and with Lipschitz bound L. By a linear map, we mean a (constant speed) geodesic. Let $G \subset \Lambda$ denote the set of immersed closed geodesics in M of length at most $2\pi L$. (The energy of a curve in Λ is equal to its length squared divided by 2π. In other words, energy and length are essentially equivalent.) Note that Λ is finite dimensional with the dimension given in terms of L and the dimension of M.

We will use the distance and topology on Λ given by the $W^{1,2}$ norm on the space of maps from $\mathbf{S}^1$ to M.

The curve shortening is a map $\Psi : \Lambda \to \Lambda$ so that[1]:

(1) $\Psi(\gamma)$ is homotopic to γ and $\text{Length}(\Psi(\gamma)) \leq \text{Length}(\gamma)$.

(2) $\Psi(\gamma)$ depends continuously on γ.

(3) There is a continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ so that

$$\text{dist}^2(\gamma, \Psi(\gamma)) \leq \phi\left(\frac{\text{Length}^2(\gamma) - \text{Length}^2(\Psi(\gamma))}{\text{Length}^2(\Psi(\gamma))}\right) .$$

(4) Given $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ with $\text{dist}(\gamma, G) \geq \epsilon$, then $\text{Length}\,(\Psi(\gamma)) \leq \text{Length}\,(\gamma) - \delta$.

Versions of these properties were known to Birkhoff, but the strong forms that we give here are proven in [**CM28**].

2.2. Defining the curve shortening Ψ. To define Ψ, we will fix a partition of $\mathbf{S}^1$ by choosing $2L$ consecutive evenly spaced points[2]

$$x_0, x_1, x_2, \ldots, x_{2L} = x_0 \in \mathbf{S}^1 \,, \tag{5.2}$$

[1]This map is essentially what is usually called Birkhoff's curve shortening process; see section 2 of [**Cr**] or [**CM28**].

[2]Note that this is not necessarily where the piecewise linear maps have breaks.

so that $|x_j - x_{j+1}| = \frac{\pi}{L}$. $\Psi(\gamma)$ is given in three steps. First, we apply step 1 to γ to get a curve γ_e, then we apply step 2 to γ_e to get a curve γ_o. In the third and final step, we reparametrize γ_o to get $\Psi(\gamma)$.

Step 1: Replace γ on each *even* interval, i.e., $[x_{2j}, x_{2j+2}]$, by the linear map with the same endpoints to get a piecewise linear curve $\gamma_e : \mathbf{S}^1 \to M$. Namely, for each j, we let $\gamma_e\big|_{[x_{2j},x_{2j+2}]}$ be the unique shortest (constant speed) geodesic from $\gamma(x_{2j})$ to $\gamma(x_{2j+2})$.

Step 2: Replace γ_e on each *odd* interval by the linear map with the same endpoints to get the piecewise linear curve $\gamma_o : \mathbf{S}^1 \to M$.

Step 3: Reparametrize γ_o (fixing $\gamma_o(x_0)$) to get the desired constant speed curve $\Psi(\gamma) : \mathbf{S}^1 \to M$.

It is easy to see that Ψ maps Λ to Λ and has property (1); cf. section 2 of [**Cr**].

To prove (2) and (3), it will be useful to observe that there is an equivalent, but more symmetric, way to construct $\Psi(\gamma)$ using four steps:

(A_1) Follow step 1 to get γ_e.

(B_1) Reparametrize γ_e (fixing the image of x_0) to get the constant speed curve $\tilde{\gamma}_e$. This reparametrization moves the points x_j to new points $\tilde{x}_j$ (i.e., $\gamma_e(x_j) = \tilde{\gamma}_e(\tilde{x}_j)$).

(A_2) Do linear replacement on the odd $\tilde{x}_j$ intervals to get $\tilde{\gamma}_o$.

(B_2) Reparametrize $\tilde{\gamma}_o$ (fixing the image of x_0) to get the constant speed curve $\Psi(\gamma)$.

The reason that this gives the same curve is that $\tilde{\gamma}_o$ is just a reparametrization of γ_o. We will also use that each of the four steps is energy non-increasing. This is obvious for the linear replacements, since linear maps minimize energy. It follows from the Cauchy-Schwarz inequality for the reparametrizations, since for a curve $\sigma : \mathbf{S}^1 \to M$ we have

$$\text{Length}^2(\sigma) \leq 2\pi\, \mathrm{E}(\sigma)\,, \tag{5.3}$$

with equality if and only if $|\sigma'| = \text{Length}(\sigma)/(2\pi)$ almost everywhere.

2.3. Property (3) for Ψ. Using the alternative way of defining $\Psi(\gamma)$ in four steps, we see that (3) follows from the triangle inequality once we bound $\text{dist}(\gamma, \gamma_e)$ and $\text{dist}(\gamma_e, \tilde{\gamma}_e)$ in terms of the decrease in length (as well as the analogs for steps (A_2) and (B_2)).

The bound on $\text{dist}(\gamma, \gamma_e)$ follows directly from the following lemma:

Lemma 5.1. *There exists C so that if I is an interval of length at most $2\pi/L$, $\sigma_1 : I \to M$ is a Lipschitz curve with $|\sigma_1'| \leq L$, and $\sigma_2 : I \to M$ is*

the minimizing geodesic with the same endpoints, then

$$\operatorname{dist}^2(\sigma_1, \sigma_2) \leq C\,(E(\sigma_1) - E(\sigma_2))\,. \tag{5.4}$$

We will need a simple consequence of (M1) and (M3):

Lemma 5.2. *If $x, y \in M$, then $\left|(x-y)^{\perp}\right| \leq |x-y|^2$, where $(x-y)^{\perp}$ is the normal component to M at y.*

Proof. If $|x-y| \geq 1$, then the claim is clear. Assume therefore that

$$|x-y| < 1$$

and $\alpha : [0, \ell] \to M$ is a minimizing unit speed geodesic from y to x with $\ell \leq 2\,|x-y|$. Let V be the unit normal vector

$$V = \frac{(x-y)^{\perp}}{|(x-y)^{\perp}|}\,,$$

so $\langle \alpha'(0), V\rangle = 0$, and observe that

$$\begin{aligned} |(x-y)^{\perp}| &= \int_0^{\ell} \langle \alpha'(s), V\rangle\, ds = \int_0^{\ell} \langle \alpha'(0) + \int_0^{s} \alpha''(t)\, dt\,, V\rangle\, ds \\ &\leq \int_0^{\ell}\int_0^{s} \left|\alpha''(t)\right|\, dt\, ds \leq \int_0^{\ell}\int_0^{s} |A(\alpha(t))|\, dt\, ds \\ &\leq \frac{1}{2}\,\ell^2 \sup_M |A| \leq |x-y|^2\,. \end{aligned} \tag{5.5}$$

□

Proof of Lemma 5.1. Integrating by parts and using that σ_1 and σ_2 are equal on ∂I gives

$$\int_I |\sigma_1'|^2 - \int_I |\sigma_2'|^2 - \int_I \left|(\sigma_1 - \sigma_2)'\right|^2 = -2\int_I \langle(\sigma_1 - \sigma_2), \sigma_2''\rangle \equiv \kappa\,. \tag{5.6}$$

The lemma will follow by bounding $|\kappa|$ by $\frac{1}{2}\int_I |(\sigma_1 - \sigma_2)'|^2$ and appealing to Wirtinger's inequality.

Since σ_2 is a geodesic on M, σ_2'' is normal to M and $|\sigma_2''| \leq |\sigma_2'|^2 \sup_M |A| \leq \frac{|\sigma_2'|^2}{16}$. Thus, Lemma 5.2 gives

$$\left|\langle(\sigma_1 - \sigma_2), \sigma_2''\rangle\right| \leq |(\sigma_1 - \sigma_2)^{\perp}|\,\frac{|\sigma_2'|^2}{16} \leq |\sigma_1 - \sigma_2|^2\,\frac{|\sigma_2'|^2}{16}\,. \tag{5.7}$$

Integrating (5.7), using that $|\sigma_2'|$ is constant with $|\sigma_2'|\,\mathrm{Length}(I) \leq 2\pi$, and applying Wirtinger's inequality gives

$$\begin{aligned} |\kappa| &\leq \frac{|\sigma_2'|^2}{8}\int_I |\sigma_1 - \sigma_2|^2 \leq \frac{|\sigma_2'|^2}{8}\left(\frac{\mathrm{Length}(I)}{\pi}\right)^2 \int_I |(\sigma_1 - \sigma_2)'|^2 \\ &\leq \frac{1}{2}\int_I \left|(\sigma_1 - \sigma_2)'\right|^2\,. \end{aligned} \tag{5.8}$$

□

Applying Lemma 5.1 on each of the L intervals in step (A_1) gives

$$\begin{aligned}\operatorname{dist}^2(\gamma,\gamma_e) &\leq C\ (\mathrm{E}(\gamma)-\mathrm{E}(\gamma_e))\\ &\leq \frac{C}{2\pi}\left(\mathrm{Length}^2(\gamma)-\mathrm{Length}^2(\Psi(\gamma))\right)\ .\end{aligned} \tag{5.9}$$

This gives the desired bound on $\operatorname{dist}(\gamma,\gamma_e)$ since $\mathrm{Length}(\Psi(\gamma))\leq 2\pi\, L$.

In bounding $\operatorname{dist}(\gamma_e,\tilde{\gamma}_e)$, we will use that γ_e is just the composition $\tilde{\gamma}_e\circ P$, where $P:\mathbf{S}^1\to\mathbf{S}^1$ is a monotone piecewise linear map.[3] Using that $|\tilde{\gamma}_e'|=\mathrm{Length}(\tilde{\gamma}_e)/(2\pi)$ (away from the breaks) and that the integral of P' is 2π, an easy calculation gives

$$\begin{aligned}\int (P'-1)^2 &= \int (P')^2-2\pi=\int\left(\frac{|\gamma_e'|}{|\tilde{\gamma}_e'\circ P|}\right)^2-2\pi\\ &= \frac{4\pi^2}{\mathrm{Length}^2(\tilde{\gamma}_e)}\int|\gamma_e'|^2-2\pi=2\pi\,\frac{\mathrm{E}(\gamma_e)-\mathrm{E}(\tilde{\gamma}_e)}{\mathrm{E}(\tilde{\gamma}_e)}\\ &\leq 2\pi\,\frac{\mathrm{E}(\gamma)-\mathrm{E}(\Psi(\gamma))}{\mathrm{E}(\Psi(\gamma))}\,.\end{aligned} \tag{5.10}$$

Since γ_e and $\tilde{\gamma}_e$ agree at $x_0=x_{2L}$, the Wirtinger inequality bounds $\operatorname{dist}^2(\gamma_e,\tilde{\gamma}_e)$ in terms of

$$\frac{1}{2}\int\left|(\tilde{\gamma}_e\circ P)'-\tilde{\gamma}_e'\right|^2\leq\int\left|(\tilde{\gamma}_e'\circ P)\,P'-\tilde{\gamma}_e'\circ P\right|^2+\left|\tilde{\gamma}_e'\circ P-\tilde{\gamma}_e'\right|^2\,. \tag{5.11}$$

We will bound both terms on the right-hand side of (5.11) in terms of $\int|P'-1|^2$ and then appeal to (5.10). To bound the first term, use that $|\tilde{\gamma}_e'|$ is (a constant) $\leq L$ to get

$$\int\left|(\tilde{\gamma}_e'\circ P)\,P'-\tilde{\gamma}_e'\circ P\right|^2\leq L^2\int|P'-1|^2\,. \tag{5.12}$$

To bound the second integral, we will use that when x and y are points in $\mathbf{S}^1$ that are *not* separated by a break point, then $\tilde{\gamma}_e$ is a geodesic from x to y and, thus, $\tilde{\gamma}_e''$ is normal to M and

$$|\tilde{\gamma}_e''|\leq|\tilde{\gamma}_e'|^2\sup_M|A|\leq\frac{L^2}{16}\,.$$

Therefore, integrating $\tilde{\gamma}_e''$ from x to y gives

$$|\tilde{\gamma}_e'(x)-\tilde{\gamma}_e'(y)|\leq|x-y|\,\sup|\tilde{\gamma}_e''|\leq\frac{L^2}{16}\,|x-y|\,. \tag{5.13}$$

Divide $\mathbf{S}^1$ into two sets, S_1 and S_2, where S_1 is the set of points within distance $(\pi\int|P'-1|^2)^{1/2}$ of a break point for $\tilde{\gamma}_e$. Since $P(x_0)=x_0$, arguing

[3] The map P is Lipschitz, but the inverse map P^{-1} may not be if γ_e is constant on an interval.

as in (5.1) gives

$$|P(x) - x| \leq \left(\pi \int |P' - 1|^2\right)^{1/2} .$$

Thus, if $x \in S_2$, then $\tilde{\gamma}_e$ is smooth between x and $P(x)$. Consequently, (5.13) gives

$$(5.14) \qquad \int_{S_2} |\tilde{\gamma}_e' \circ P - \tilde{\gamma}_e'|^2 \leq \frac{L^4}{256} \int_{S_2} |P(s) - s|^2 \leq \frac{L^4}{64} \int |P' - 1|^2 ,$$

where the last inequality used the Wirtinger inequality. On the other hand,

$$(5.15) \quad \int_{S_1} |\tilde{\gamma}_e' \circ P - \tilde{\gamma}_e'|^2 \leq 4\, L^2 \; \text{Length}(S_1) \leq 8\, L^3 \left(\pi \int |P' - 1|^2\right)^{1/2} ,$$

completing the proof of property (3).

2.4. Property (4). To prove property (4), we will argue by contradiction. Suppose therefore that there exist $\epsilon > 0$ and a sequence $\gamma_j \in \Lambda$ with

$$(5.16) \qquad \mathrm{E}(\Psi(\gamma_j)) \geq \mathrm{E}(\gamma_j) - 1/j ,$$

$$(5.17) \qquad \mathrm{dist}(\gamma_j, G) \geq \epsilon > 0 .$$

Note that the second condition implies a positive lower bound for $\mathrm{E}(\gamma_j)$. Observe next that the space Λ is compact since $\sigma \in \Lambda$ depends continuously on the images of the L break points in the compact manifold M. Therefore, a subsequence of the γ_j's must converge to some $\gamma \in \Lambda$. Since property (3) implies that

$$\mathrm{dist}(\gamma_j, \Psi(\gamma_j)) \to 0 ,$$

the $\Psi(\gamma_j)$'s also converge to γ. The continuity of Ψ, i.e., property (2) of Ψ, then implies that $\Psi(\gamma) = \gamma$. However, this implies that $\gamma \in G$ since the only fixed points of Ψ are immersed closed geodesics. This last fact, which was used already by Birkhoff (see section 2 in [**Cr**]), follows immediately from Lemma 5.1 and (5.10). However, this would contradict that the γ_j's remain a fixed distance from any such closed immersed geodesic, completing the proof of (4).

2.5. Property (2): The continuity of Ψ.

Lemma 5.3. *Let $\gamma : \mathbf{S}^1 \to M$ be a $W^{1,2}$ map with $E(\gamma) \leq L$. If γ_e and $\tilde{\gamma}_e$ are given by applying steps (A_1) and (B_1) to γ, then the map $\gamma \to \tilde{\gamma}_e$ is continuous from $W^{1,2}$ to Λ equipped with the $W^{1,2}$ norm.*

Proof. It follows from (5.1) and the energy bound that

$$(5.18) \qquad \mathrm{dist}_M(\gamma(x_{2j}), \gamma(x_{2j+2})) \leq 2\pi$$

for each j and thus we can apply step (A_1). The lemma will follow easily from two observations:

(C1) Since $W^{1,2}$ close curves are also C^0 close, it follows that the points $\gamma_e(x_{2j}) = \gamma(x_{2j})$ are continuous with respect to the $W^{1,2}$ norm.

(C2) Define $\Gamma \subset M \times M$ by

$$\Gamma = \{(x, y) \in M \times M \mid \text{dist}_M(x, y) \leq 4\pi\},$$

and define a map

$$H : \Gamma \to C^1([0, 1], M)$$

by letting $H(x, y) : [0, 1] \to M$ be the linear map from x to y. Then the map H is continuous on Γ. Furthermore, the map $t \to H(x, y)(t)$ has uniformly bounded first and second derivatives

$$|\partial_t H(x, y)| \leq 4\pi,$$
$$|\partial_t^2 H(x, y)| \leq \pi^2.$$

The second derivative bound comes from (M1).[4]

To prove the lemma, suppose that γ^1 and γ^2 are nonconstant curves in Λ (continuity at the constant maps is obvious). For $i = 1, 2$ and $j = 1, \ldots, L$, let a_j^i be the distance in M from $\gamma^i(x_{2j})$ to $\gamma^i(x_{2j+2})$. Let $S^i = \frac{1}{2\pi} \sum_{j=1}^{L} a_j^i$ be the speed of $\tilde{\gamma}_e^i$, so that $|(\tilde{\gamma}_e^i)'| = S^i$ except at the L break points. By (C1), the a_j^i's are continuous functions of γ^i and, thus, so are S^1 and S^2. Moreover, (C1) and (C2) imply that γ_e^1 and γ_e^2 are C^1-close on each interval $[x_{2j}, x_{2j+2}]$. Thus, we have shown that $\gamma \to \gamma_e$ is continuous.

To show that $\gamma_e \to \tilde{\gamma}_e$ is also continuous, we will show that the $\tilde{\gamma}_e^i$'s are close when the γ_e^i's are. Since the point $x_0 = x_{2L}$ is fixed under the reparametrization, this will follow from applying Wirtinger's inequality to $(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2) - (\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)(x_0)$ once we show that $\int_{\mathbf{S}^1} |(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'|^2$ can be made small.

The piecewise linear curve $\tilde{\gamma}_e^i$ is linear on the intervals

$$I_j^i = \left[\frac{1}{S^i} \sum_{\ell < j} a_\ell^i, \frac{1}{S^i} \sum_{\ell \leq j} a_\ell^i\right]. \tag{5.19}$$

Set $I_j = I_j^1 \cap I_j^2$. Observe first that since the intervals I_j^i in (5.19) depend continuously on γ_e^i, the measure of the complement $\mathbf{S}^1 \setminus \left[\bigcup_{j=1}^L I_j\right]$ can be made small, so that

$$\int_{\mathbf{S}^1 \setminus [\bigcup I_j]} \left|(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'\right|^2 \leq 4\, L^2 \text{ Length} \left(\mathbf{S}^1 \setminus \left[\bigcup I_j\right]\right) \tag{5.20}$$

[4] (M1) is defined in the beginning of the chapter.

can also be made small. We will divide the I_j's into two groups, depending on the size of a_j^1. Fix some $\epsilon > 0$ and suppose first that $a_j^1 < \epsilon$; by continuity, we can assume that $a_j^2 < 2\epsilon$. For such a j, we get

$$\begin{aligned} \int_{I_j} \left|(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'\right|^2 &\leq 2 \int_{I_j^1} \left|(\tilde{\gamma}_e^1)'\right|^2 + 2 \int_{I_j^2} \left|(\tilde{\gamma}_e^2)'\right|^2 \\ &\leq 2\,L\,\left(a_j^1 + a_j^2\right) \leq 6\,\epsilon\,L\,. \end{aligned} \tag{5.21}$$

Since there are at most L breaks, summing over these intervals contributes at most $6\epsilon\, L^2$ to the energy of $(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)$.

The last case to consider is an I_j with $a_j^1 \geq \epsilon$; by continuity, we can assume that $a_j^2 \geq \epsilon/2$. In this case, $\tilde{\gamma}_e^i$ can be written on I_j as the composition $\gamma_e^i \circ P_j^i$ where $\left|(P_j^i)'\right| = 2\pi\, S^i/(La_j^i)$. Furthermore, P_j^1 and P_j^2 both map I_j into $[x_{2j}, x_{2j+2}]$ and

$$\int_{I_j} \left|(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'\right|^2 = \int_{I_j} \left|(\gamma_e^1 \circ P_j^1 - \gamma_e^2 \circ P_j^2)'\right|^2 . \tag{5.22}$$

Finally, this can be made small since the speed $\left|(P_j^i)'\right|$ is continuous[5] in γ^i and the γ_e^i's are C^2 bounded and C^1 close on $[x_{2j}, x_{2j+2}]$. Therefore, the integral over these intervals can also be made small since there are at most L of them. □

3. Existence of Closed Geodesics and the Width

Let Ω be the set of continuous maps

$$\sigma : \mathbf{S}^1 \times [0,1] \to M$$

so that:

- For each t the map $\sigma(\cdot, t)$ is in $W^{1,2}$.
- The map $t \to \sigma(\cdot, t)$ is continuous from $[0,1]$ to $W^{1,2}$.
- σ maps $\mathbf{S}^1 \times \{0\}$ and $\mathbf{S}^1 \times \{1\}$ to points.

Given a map $\hat{\sigma} \in \Omega$, the homotopy class $\Omega_{\hat{\sigma}}$ is defined to be the set of maps $\sigma \in \Omega$ that are homotopic to $\hat{\sigma}$ through maps in Ω. The width $W = W(\hat{\sigma})$ associated to the homotopy class $\Omega_{\hat{\sigma}}$ is defined by taking inf of max of the energy of each slice. That is, set

$$W = \inf_{\sigma \in \Omega_{\hat{\sigma}}} \max_{t \in [0,1]} \mathrm{E}\,(\sigma(\cdot, t))\,, \tag{5.23}$$

where the energy is given by $\mathrm{E}\,(\sigma(\cdot,t)) = \int_{\mathbf{S}^1} \left|\partial_x \sigma(x,t)\right|^2 dx$. The width is always nonnegative and is positive if $\hat{\sigma}$ is in a nontrivial homotopy class.

[5] The speed is continuous because of the lower bound for the a_j^i's.

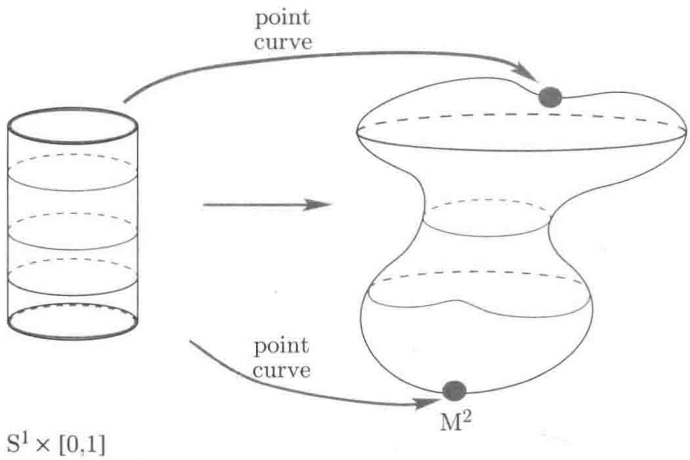

Figure 5.1. A sweepout.

A particularly interesting example is when M is a topological 2-sphere and the induced map from $\mathbf{S}^2$ to M has degree one. In this case, the width is positive and realized by a nontrivial closed geodesic. To see that the width is positive on nontrivial homotopy classes, observe that if the maximal energy of a slice is sufficiently small, then each curve $\sigma(\cdot, t)$ is contained in a convex geodesic ball in M. Hence, a geodesic homotopy connects σ to a path of point curves, so σ is homotopically trivial.

The main theorem of this section, Theorem 5.6, of which almost maximal slices in the tightened sweepouts are almost geodesics, is proven in subsection 3.4. The proof of this theorem as well as the construction of the sequence of tighter and tighter sweepouts uses a curve shortening map that is defined in the next subsection.

3.1. The width is continuous in the metric. It is not hard to see that the width is continuous in the metric, but the min-max curve that realizes it may not be; see the example below. In fact, elaborating on this example one can easily see that the width is not in general more than continuous in the metric.

Lemma 5.4. *The width is continuous on a smooth one-parameter family of metrics $\{g_t\}_{t\in[0,1]}$ on a fixed closed surface.*

Proof. The continuity follows immediately from the following:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $t \in [0,1]$ and $|s-t| < \delta$, then

$$W(g_s) < W(g_t) + \epsilon\,. \qquad \square$$

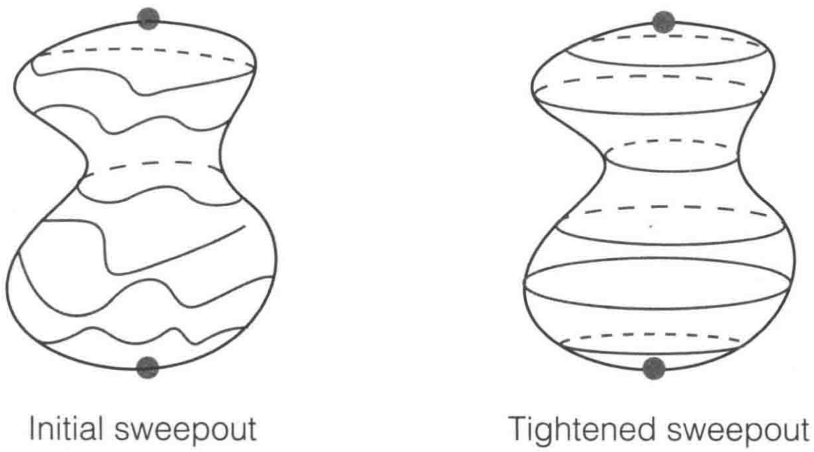

Figure 5.2. Birkhoff's idea of pulling the sweepout tight to obtain a closed geodesic.

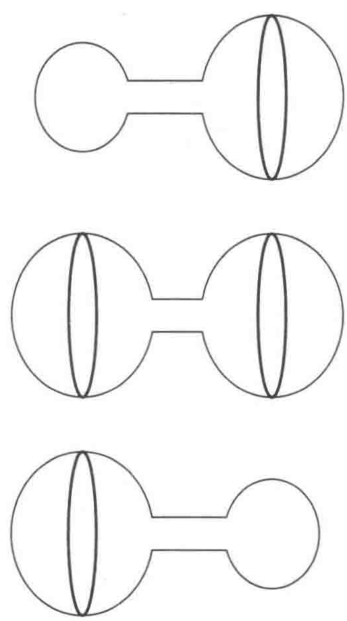

Figure 5.3. Three snapshots of a one-parameter family of "dumbbell" metrics. The geodesic realizing the width jumps from one bell to the other. The jump occurs in the middle picture where the geodesic is not unique.

3.2. The curve shortening and sweepouts. The next result shows that Ψ preserves the homotopy class of a sweepout.

Lemma 5.5. *Let* $\gamma \in \Omega$ *satisfy* $\max_t \, E(\gamma(\cdot, t)) \leq L$. *If* γ_e *and* $\tilde{\gamma}_e$ *are given by applying steps* (A_1) *and* (A_2) *to each* $\gamma(\cdot, t)$, *then* γ, γ_e *and* $\tilde{\gamma}_e$ *are all homotopic in* Ω.

Proof. Given $x, y \in M$ with $\text{dist}_M(x, y) \leq 4\pi$, let $H(x, y) : [0, 1] \to M$ be the linear map from x to y as in (C2). It follows that

$$F(x, t, s) = H(\gamma(x, t), \gamma_e(x, t))(s) \tag{5.24}$$

is an explicit homotopy with $F(\cdot, \cdot, 0) = \gamma$ and $F(\cdot, \cdot, 1) = \gamma_e$.

For each t with $\text{Length}(\gamma_e(\cdot, t)) > 0$, γ_e is given by $\gamma_e(\cdot, t) = \tilde{\gamma}_e(\cdot, t) \circ P_t$ where P_t is a monotone reparametrization of $\mathbf{S}^1$ that fixes $x_0 = x_{2L}$. Moreover, P_t is continuous by (5.10) and P_t depends continuously on t by Lemma 5.3. Since $x \to (1 - s)P_t(x) + sx$ gives a homotopy from P_t to the identity map on $\mathbf{S}^1$, we conclude that

$$G(x, t, s) = \tilde{\gamma}_e\left((1 - s)P_t(x) + sx, t\right) \tag{5.25}$$

is an explicit homotopy with $G(\cdot, \cdot, 0) = \gamma_e$ and $G(\cdot, \cdot, 1) = \tilde{\gamma}_e$. Note that P_t is not defined when $\text{Length}(\gamma_e(\cdot, t)) = 0$, but the homotopy G is. $\square$

3.3. Defining the sweepouts. Choose a sequence of maps $\hat{\sigma}^j \in \Omega_{\hat{\sigma}}$ with

$$\max_{t \in [0,1]} \text{E}\left(\hat{\sigma}^j(\cdot, t)\right) < W + \frac{1}{j}. \tag{5.26}$$

Observe that (5.26) and the Cauchy-Schwarz inequality imply a uniform bound for the length and uniform $C^{1/2}$ continuity for the slices, that are both independent of t and j. The first follows immediately and the latter follows from

$$\begin{aligned} \left|\hat{\sigma}^j(x, t) - \hat{\sigma}^j(y, t)\right|^2 &\leq \left(\int_x^y \left|\partial_s \hat{\sigma}^j(s, t)\right| ds\right)^2 \\ &\leq |y - x| \int_x^y \left|\partial_s \hat{\sigma}^j(s, t)\right|^2 ds \leq |y - x| \left(W + 1\right). \end{aligned} \tag{5.27}$$

We will replace the $\hat{\sigma}^j$'s by sweepouts σ^j that, in addition to satisfying (5.26), also satisfy that the slices $\sigma^j(\cdot, t)$ are in Λ. We will do this by using local linear replacement similar to step 1 of the construction of Ψ. Namely, the uniform $C^{1/2}$ bound for the slices allows us to fix a partition of points $y_0, \ldots, y_N = y_0$ in $\mathbf{S}^1$ so that each interval $[y_i, y_{i+1}]$ is always mapped to a ball in M of radius at most 4π. Next, for each t and each j, we replace $\hat{\sigma}^j(\cdot, t)\big|_{[y_i, y_{i+1}]}$ by the linear map (geodesic) with the same endpoints and call the resulting map $\tilde{\sigma}^j(\cdot, t)$. Reparametrize $\tilde{\sigma}^j(\cdot, t)$ to have constant speed to get $\sigma^j(\cdot, t)$. It is easy to see that each $\sigma^j(\cdot, t)$ satisfies (5.26). Furthermore, the length bound for $\sigma^j(\cdot, t)$ also gives a uniform Lipshitz bound for the linear maps; let L be the maximum of N and this Lipshitz bound.

It remains to check that σ^j is continuous in the transversal direction, i.e., with respect to t, and homotopic to $\hat{\sigma}$ in Ω.[6] However, this follows immediately from Lemmas 5.3 and 5.5.

Finally, applying the replacement map Ψ to each $\sigma^j(\cdot, t)$ gives a new sequence of sweepouts $\gamma^j = \Psi(\sigma^j)$. (By Lemmas 5.3 and 5.5, Ψ depends continuously on t and preserves the homotopy class $\Omega_{\hat{\sigma}}$; it is clear that Ψ fixes the constant maps at $t = 0$ and $t = 1$.)

3.4. Almost maximal implies almost critical. We show the following useful property; see Theorem 5.5 below and cf. [CM19], [CM27], proposition 3.1 of [CD], proposition 3.1 of [Pi], and 12.5 of [Am3]:

> Each curve in the tightened sweepout whose length is close to the length of the longest curve in the sweepout must itself be close to a closed geodesic. In particular, there are curves in the sweepout that are close to closed geodesics.

The above useful property is virtually always implicit in any sweepout construction of critical points for variational problems yet it is not always recorded since most authors are only interested in the existence of one critical point.

As an application, we show that this sequence γ^j of sweepouts is tight. Namely, we have the following theorem of [**CM28**]:

Theorem 5.6 (Colding-Minicozzi, [**CM28**]). *Given $W \geq 0$ and $\epsilon > 0$, there exist $\delta > 0$ so that if $j > 1/\delta$ and for some t_0,*

$$2\pi\, E(\gamma^j(\cdot, t_0)) = \text{Length}^2\,(\gamma^j(\cdot, t_0)) > 2\pi\,(W - \delta)\,, \tag{5.28}$$

then for this j we have $\text{dist}\,\big(\gamma^j(\cdot, t_0)\,,\, G\big) < \epsilon$.

As an immediate consequence, we get the existence of nontrivial closed geodesics for any metric on $\mathbf{S}^2$; this is due to Birkhoff. See [**LzWl**] for an alternative proof using the harmonic map heat flow.

The next lemma, which combines properties (3) and (4) of the curve shortening Ψ, is the key to producing the desired sequence of sweepouts.

Lemma 5.7. *Given $W \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ and*

$$2\pi\,(W - \delta) < \text{Length}^2\,(\Psi(\gamma)) \leq \text{Length}^2\,(\gamma) < 2\pi\,(W + \delta)\,, \tag{5.29}$$

then $\text{dist}(\Psi(\gamma), G) < \epsilon$.

[6] These facts were established already by Birkhoff (see [**B1**], [**B2**] and section 2 of [**Cr**]).

Proof. If $W \leq \epsilon^2/6$, the Wirtinger inequality gives the lemma with $\delta = \epsilon^2/6$.

Assume next that $W > \epsilon^2/6$. The triangle inequality gives

$$\text{dist}(\Psi(\gamma), G) \leq \text{dist}(\Psi(\gamma), \gamma) + \text{dist}(\gamma, G)\,. \tag{5.30}$$

Since Ψ does not decrease the length of γ by much, property (4) of Ψ allows us to bound $\text{dist}(\gamma, G)$ by $\epsilon/2$ as long as δ is sufficiently small. Similarly, property (3) of Ψ allows us to bound $\text{dist}(\Psi(\gamma), \gamma)$ by $\epsilon/2$ as long as δ is sufficiently small. $\square$

Proof of Theorem 5.6. Let δ be given by Lemma 5.7. By (5.28), (5.26), and using that $j > 1/\delta$, we get

$$2\pi\,(W - \delta) < \text{Length}^2\,(\gamma^j(\cdot, t_0)) \leq \text{Length}^2\,(\sigma^j(\cdot, t_0)) < 2\pi\,(W + \delta)\,. \tag{5.31}$$

Thus, since $\gamma^j(\cdot, t_0) = \Psi(\sigma^j(\cdot, t_0))$, Lemma 5.7 gives $\text{dist}(\gamma^j(\cdot, t_0)\,,\, G) < \epsilon$, as claimed. $\square$

3.5. Parameter spaces. Instead of using the unit interval, $[0, 1]$, as the parameter space for the circles in the sweepout and assuming that the curves start and end in point curves, we could have used any compact set $\mathcal{P}$ and required that the curves are constant on $\partial\mathcal{P}$ (or that $\partial\mathcal{P} = \emptyset$). In this case, let $\Omega^{\mathcal{P}}$ be the set of continuous maps $\sigma : \mathbf{S}^1 \times \mathcal{P} \to M$ so that for each $t \in \mathcal{P}$ the curve $\sigma(\cdot, t)$ is in $W^{1,2}$, the map $t \to \sigma(\cdot, t)$ is continuous from $\mathcal{P}$ to $W^{1,2}$, and finally σ maps $\partial\mathcal{P}$ to point curves. Given a map $\hat{\sigma} \in \Omega^{\mathcal{P}}$, the homotopy class $\Omega^{\mathcal{P}}_{\hat{\sigma}} \subset \Omega^{\mathcal{P}}$ is defined to be the set of maps $\sigma \in \Omega^{\mathcal{P}}$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega^{\mathcal{P}}$. Finally, the width $W = W(\hat{\sigma})$ is

$$W = \inf_{\sigma \in \Omega^{\mathcal{P}}_{\hat{\sigma}}} \max_{t \in \mathcal{P}} \text{E}\,(\sigma(\cdot, t))\,. \tag{5.32}$$

Theorem 5.6 holds for these general parameter spaces; the proof is virtually the same with only trivial changes.

4. Harmonic Replacement

We turn next to maps from $\mathbf{S}^2$. The rough idea is the same as for curves: We define an energy decreasing "discrete gradient flow" and show that it strictly decreases energy away from harmonic maps. The energy decreasing map will again be constructed by local replacement, but this time we use replacement by harmonic (energy-minimizing) maps on balls with small energy. This is often referred to as *harmonic replacement* and is reminiscent of H.A. Schwarz's alternating method for solving the Dirichlet problem for harmonic functions.

4.1. The properties of harmonic replacement. In this subsection, we will summarize the key properties of harmonic replacement. These properties were proven in [**CM27**] and we refer to the original paper for their proofs.

The first key property that makes harmonic replacement useful is that the energy functional is strictly convex on small energy maps. This is a *convexity theorem for harmonic maps.* The precise statement of this is given by the following result of Colding-Minicozzi, [**CM27**]:

Theorem 5.8 (Colding-Minicozzi, [**CM27**]). *There exists a constant $\epsilon_1 > 0$ (depending on M) so that if u and v are $W^{1,2}$ maps from $B_1 \subset \mathbb{R}^2$ to M, u and v agree on ∂B_1, and v is weakly harmonic with energy at most ϵ_1, then*

$$\int_{B_1} |\nabla u|^2 - \int_{B_1} |\nabla v|^2 \geq \frac{1}{2} \int_{B_1} |\nabla v - \nabla u|^2 . \tag{5.33}$$

An immediate corollary of Theorem 5.8 is uniqueness of solutions to the Dirichlet problem for small energy maps (and also that any such harmonic map minimizes energy). This theorem relies upon estimates of Hélein, [**Hf1**] and [**Hf2**], for weakly harmonic maps as well as compensated compactness techniques dating back to Wente's work in [**W**].

Corollary 5.9 (Colding-Minicozzi, [**CM27**]). *Let $\epsilon_1 > 0$ be as in Theorem 5.8. If u_1 and u_2 are $W^{1,2}$ weakly harmonic maps from $B_1 \subset \mathbb{R}^2$ to M, both with energy at most ϵ_1, and they agree on ∂B_1, then $u_1 = u_2$.*

The second consequence of Theorem 5.8 is that harmonic replacement is continuous as a map from $C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ to itself if we restrict to small energy maps. (The norm on $C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ is the sum of the sup norm and the $W^{1,2}$ norm.)

Corollary 5.10. *Let $\epsilon_1 > 0$ be as in Theorem 5.8 and set*

$$\mathcal{M} = \{u \in C^0(\overline{B_1}, M) \cap W^{1,2}(B_1, M) \mid E(u) \leq \epsilon_1\} . \tag{5.34}$$

Given $u \in \mathcal{M}$, there is a unique energy minimizing map w equal to u on ∂B_1 and w is in $\mathcal{M}$. Furthermore, there exists C depending on M so that if $u_1, u_2 \in \mathcal{M}$ with corresponding energy minimizing maps w_1, w_2, and we set $E = E(u_1) + E(u_2)$, then

$$|E(w_1) - E(w_2)| \leq C \, ||u_1 - u_2||_{C^0(\overline{B_1})} \, E + C \, ||\nabla u_1 - \nabla u_2||_{L^2(B_1)} \, E^{1/2} . \tag{5.35}$$

Finally, the map from u to w is continuous as a map from $C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ to itself.

It will be convenient to introduce some notation for the next lemma. Namely, given a $C^0 \cap W^{1,2}$ map u from $\mathbf{S}^2$ to M and a finite collection $\mathcal{B}$ of disjoint closed balls in $\mathbf{S}^2$ so the energy of u on $\bigcup_{\mathcal{B}} B$ is at most $\epsilon_1/3$,

let $H(u,\mathcal{B}) : \mathbf{S}^2 \to M$ denote the map that coincides with u on $\mathbf{S}^2 \setminus \bigcup_{\mathcal{B}} B$ and on $\bigcup_{\mathcal{B}} B$ is equal to the energy minimizing map from $\bigcup_{\mathcal{B}} B$ to M that agrees with u on $\bigcup_{\mathcal{B}} \partial B$. To keep the notation simple, we will set

$$H(u,\mathcal{B}_1,\mathcal{B}_2) = H(H(u,\mathcal{B}_1),\mathcal{B}_2)\,. \tag{5.36}$$

Finally, if $\alpha \in (0,1]$, then $\alpha\mathcal{B}$ will denote the collection of concentric balls but whose radii are shrunk by the factor α.

In general, $H(u,\mathcal{B}_1,\mathcal{B}_2)$ is not the same as $H(u,\mathcal{B}_2,\mathcal{B}_1)$. This matters in the proof of Theorem 5.23, where harmonic replacement on either $\frac{1}{2}\mathcal{B}_1$ or $\frac{1}{2}\mathcal{B}_2$ decreases the energy of u by a definite amount. The next lemma (see (5.37)) shows that the energy goes down a definite amount regardless of the order in which we do the replacements. The second inequality bounds the possible decrease in energy from applying harmonic replacement on $H(u,\mathcal{B}_1)$ in terms of the possible decrease from harmonic replacement on u.

Lemma 5.11. *There is a constant $\kappa > 0$ (depending on M) so that if $u : \mathbf{S}^2 \to M$ is in $C^0 \cap W^{1,2}$ and $\mathcal{B}_1$, $\mathcal{B}_2$ are each finite collections of disjoint closed balls in $\mathbf{S}^2$ so that the energy of u on each $\bigcup_{\mathcal{B}_i} B$ is at most $\epsilon_1/3$, then*

$$E(u) - E[H(u,\mathcal{B}_1,\mathcal{B}_2)] \geq \kappa \left(E(u) - E\left[H(u,\frac{1}{2}\mathcal{B}_2)\right]\right)^2 . \tag{5.37}$$

Furthermore, for any $\mu \in [1/8, 1/2]$, we have

$$\begin{aligned} &\frac{\left(E(u) - E[H(u,\mathcal{B}_1)]\right)^{1/2}}{\kappa} + E(u) - E[H(u, 2\,\mu\mathcal{B}_2)] \\ &\qquad \geq E[H(u,\mathcal{B}_1)] - E[H(u,\mathcal{B}_1,\mu\,\mathcal{B}_2)]\,. \end{aligned} \tag{5.38}$$

4.2. Compactness of almost harmonic maps. Our notion of almost harmonic relies on two important properties of harmonic maps from $\mathbf{S}^2$ to M. The first is that harmonic maps from $\mathbf{S}^2$ are conformal and, thus, energy and area are equal; see (A) below. The second is that any harmonic map from a surface is energy minimizing when restricted to balls where the energy is sufficiently small; see (B) below.

In the proposition, $\epsilon_{SU} > 0$ (depending on M) is the small energy constant from Theorem 4.22, so that we get interior estimates for harmonic maps with energy at most ϵ_{SU}. In particular, any nonconstant harmonic map from $\mathbf{S}^2$ to M has energy greater than ϵ_{SU}.

Proposition 5.12. *Suppose that $\epsilon_0, E_0 > 0$ are constants with $\epsilon_{SU} > \epsilon_0$ and $u^j : \mathbf{S}^2 \to M$ is a sequence of $C^0 \cap W^{1,2}$ maps with $E_0 \geq E(u^j)$ satisfying:*

(A) $\text{Area}(u^j) > E(u^j) - 1/j$.

(B) *For any finite collection $\mathcal{B}$ of disjoint closed balls in $\mathbf{S}^2$ with $\int_{\bigcup_{\mathcal{B}} B} |\nabla u^j|^2 < \epsilon_0$ there is an energy minimizing map $v : \bigcup_{\mathcal{B}} \frac{1}{8} B \to M$ that equals u^j on $\bigcup_{\mathcal{B}} \frac{1}{8} \partial B$ with*

$$\int_{\bigcup_{\mathcal{B}} \frac{1}{8} B} \left| \nabla u^j - \nabla v \right|^2 \leq 1/j \, .$$

If (A) and (B) are satisfied, then a subsequence of the u^j's varifold converges to a collection of harmonic maps $v^0, \ldots, v^m : \mathbf{S}^2 \to M$.

Varifold convergence is defined in Chapter 3.

One immediate consequence of Proposition 5.12 is a compactness theorem for sequences of harmonic maps with bounded energy. This was proven by Jost in lemma 4.3.1 in [**Jo**]. In fact, Parker proved compactness of bounded energy harmonic maps in a stronger topology, with C^0 convergence in addition to $W^{1,2}$ convergence; see theorem 2.2 in [**Pk**]. Therefore, it is perhaps not surprising that a similar compactness holds for sequences that are closer and closer to being harmonic in the sense above. However, it is useful to keep in mind that Parker has constructed sequences of maps where the Laplacian is going to zero in L^1 and yet there is no convergent subsequence (see proposition 4.2 in [**Pk**]).

Finally, we point out that Proposition 5.12 can be thought of as a discrete version of Palais-Smale Condition (C). Namely, if we have a sequence of maps where the maximal energy decrease from harmonic replacement goes to zero, then a subsequence converges to a collection of harmonic maps.

The proof of Proposition 5.12 will follow the general structure developed by Parker and Wolfson in [**PkW**] and used by Parker in [**Pk**] to prove compactness of harmonic maps with bounded energy. The main difficulty is to rule out loss of energy in the limit (see (B4) in the definition of bubble convergence; this is known as the energy identity). The rough idea to deal with this is that energy loss only occurs when there are very small annuli where the maps are "almost" harmonic and the ratio between the inner and outer radii of the annulus is enormous. We will use Proposition 5.18 to show that the map must be "far" from being conformal on such an annulus and, thus, condition (A) allows us to rule out energy loss. Here "far" from conformal will mean that the θ-energy of the map is much less than the radial energy. To make this precise, it is convenient to replace an annulus $B_{e^{r_2}} \setminus B_{e^{r_1}}$ in $\mathbb{R}^2$ by the conformally equivalent cylinder $[r_1, r_2] \times \mathbf{S}^1$. The (noncompact) cylinder $\mathbb{R} \times \mathbf{S}^1$ with the flat product metric and coordinates t and θ will be denoted by $\mathcal{C}$. For $r_1 < r_2$, let $\mathcal{C}_{r_1, r_2} \subset \mathcal{C}$ be the product $[r_1, r_2] \times \mathbf{S}^1$.

4.3. Harmonic maps on cylinders. The main result of this subsection is that harmonic maps with small energy on long cylinders are almost radial. This implies that a sequence of such maps with energy bounded away from zero is uniformly far from being conformal and, thus, cannot satisfy (A) in Proposition 5.12. It will be used to prove a similar result for "almost harmonic" maps in Proposition 5.18 and eventually be used when we show that energy will not be lost.

Proposition 5.13. *Given $\delta > 0$, there exist $\epsilon_2 > 0$ and $\ell \geq 1$ depending on δ (and M) so that if u is a (nonconstant) C^3 harmonic map from the flat cylinder $\mathcal{C}_{-3\ell,3\ell} = [-3\ell, 3\ell] \times \mathbf{S}^1$ to M with $E(u) \leq \epsilon_2$, then*

$$\int_{\mathcal{C}_{-\ell,\ell}} |u_\theta|^2 < \delta \int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2 . \tag{5.39}$$

To show this proposition, we show a differential inequality which leads to exponential growth for the θ-energy of the harmonic map on the level sets of the cylinder. Once we have that, the proposition follows. Namely, if the θ-energy in the "middle" of the cylinder was a definite fraction of the total energy over the double cylinder, then the exponential growth would force the θ-energy near the boundary of the cylinder to be too large.

The following standard lemma is the differential inequality for the θ-energy that leads to exponential growth through Lemma 5.15 below.

Lemma 5.14. *For a C^3 harmonic map u from $\mathcal{C}_{r_1,r_2} \subset \mathcal{C}$ to $M \subset \mathbb{R}^N$,*

$$\partial_t^2 \int_t |u_\theta|^2 \geq \frac{3}{2} \int_t |u_\theta|^2 - 2 \sup_M |A|^2 \int_t |\nabla u|^4 . \tag{5.40}$$

Proof. Differentiating $\int_t |u_\theta|^2$ and integrating by parts in θ gives

$$\begin{aligned}\frac{1}{2} \partial_t^2 \int_t |u_\theta|^2 &= \int_t |u_{t\theta}|^2 + \int_t \langle u_\theta, u_{tt\theta}\rangle = \int_t |u_{t\theta}|^2 - \int_t \langle u_{\theta\theta}, u_{tt}\rangle \\ &= \int_t |u_{t\theta}|^2 - \int_t \langle u_{\theta\theta}, (\Delta u - u_{\theta\theta})\rangle \\ &\geq \int_t |u_{t\theta}|^2 + \int_t |u_{\theta\theta}|^2 - \sup_M |A| \int_t |u_{\theta\theta}|\, |\nabla u|^2 , \end{aligned} \tag{5.41}$$

where the last inequality used that $|\Delta u| \leq |\nabla u|^2 \sup_M |A|$ by the harmonic map equation, (4.34). The lemma follows from applying the absorbing inequality $2ab \leq a^2/2 + 2b^2$ and noting that $\int_t u_\theta = 0$ so that Wirtinger's inequality gives $\int_t |u_\theta|^2 \leq \int_t |u_{\theta\theta}|^2$. □

We will need a simple ODE comparison lemma:

Lemma 5.15. *Suppose that f is a nonnegative C^2 function on $[-2\ell, 2\ell] \subset \mathbb{R}$ satisfying*

$$f'' \geq f - a\,, \tag{5.42}$$

for some constant $a > 0$. If $\max_{[-\ell,\ell]} f \geq 2a$, then

$$\int_{-2\ell}^{2\ell} f \geq 2\sqrt{2}\, a \sinh(\ell/\sqrt{2})\,. \tag{5.43}$$

Proof. Fix some $x_0 \in [-\ell, \ell]$ where f achieves its maximum on $[-\ell, \ell]$. Since the lemma is invariant under reflection $x \to -x$, we can assume that $x_0 \geq 0$. If x_0 is an interior point, then $f'(x_0) = 0$; otherwise, if $x_0 = \ell$, then $f'(x_0) \geq 0$. In either case, we get $f'(x_0) \geq 0$. Since $f(x_0) \geq 2a$, (5.42) gives $f''(x_0) \geq a > 0$ and, hence, f' is strictly increasing at x_0.

We claim that $f'(x) > 0$ for all x in $(x_0, 2\ell]$. If not, then there would be a first point $y > x_0$ with $f'(y) = 0$. It follows that $f' \geq 0$ on $[x_0, y]$ so that $f \geq f(x_0) \geq 2a$ on $[x_0, y]$ and, thus, that $f'' \geq a > 0$ on $[x_0, y]$, contradicting that $f'(y) \leq f'(x_0)$.

By the claim, f is monotone increasing on $[x_0, 2\ell]$ so that (5.42) gives

$$f'' \geq \frac{1}{2} f \text{ on } [x_0, 2\ell]\,. \tag{5.44}$$

By a standard Riccati comparison argument using $f'(x_0) \geq 0$ and (5.44) (see, e.g., corollary $A.9$ in [**CDM**]), we get for $t \in [0, 2\ell - x_0]$,

$$f(x_0 + t) \geq f(x_0) \cosh(t/\sqrt{2}) \geq 2\, a \cosh(t/\sqrt{2})\,. \tag{5.45}$$

Finally, integrating (5.45) on $[0, \ell]$ gives (5.43). □

Proof of Proposition 5.13. Since we will choose $\ell \geq 1$ and $\epsilon_2 < \epsilon_{SU}$, the small-energy interior estimates for harmonic maps (see Theorem 4.22) imply that

$$\sup_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2 \leq C_{SU} \int_{\mathcal{C}_{-3\ell,3\ell}} |\nabla u|^2 \leq C_{SU}\, \epsilon_2\,. \tag{5.46}$$

Set $f(t) = \int_t |u_\theta|^2$. It follows from Lemma 5.14 that

$$\begin{aligned} f''(t) &\geq \frac{3}{2} f(t) - 2 \sup_M |A|^2\, C_{SU}\, \epsilon_2 \int_t (|u_\theta|^2 + |u_t|^2) \\ &\geq f(t) - C\, \epsilon_2 \int_t (|u_t|^2 - |u_\theta|^2)\,, \end{aligned} \tag{5.47}$$

where $C = 2\, C_{SU} \sup_M |A|^2$ and we have assumed that $C\, \epsilon_2 \leq 1/4$ in the second inequality.

We will use that $\int_t (|u_t|^2 - |u_\theta|^2)$ is constant in t. To see this, differentiate to get

$$\frac{1}{2}\,\partial_t \int_t (|u_t|^2 - |u_\theta|^2) = \int_t (\langle u_t, u_{tt}\rangle - \langle u_\theta, u_{t\theta}\rangle)$$

$$= \int_t \langle u_t, (u_{tt} + u_{\theta\theta})\rangle = 0\,, \tag{5.48}$$

where the second equality used integration by parts in θ and the last equality used that $u_{tt} + u_{\theta\theta} = \Delta u$ is normal to M while u_t is tangent. Bound this constant by

$$\int_t (|u_t|^2 - |u_\theta|^2) = \frac{1}{4\ell}\int_{\mathcal{C}_{-2\ell,2\ell}} (|u_t|^2 - |u_\theta|^2) \le \frac{1}{4\ell}\int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2\,. \tag{5.49}$$

By (5.47) and (5.49), Lemma 5.15 with $a = \frac{C\,\epsilon_2}{4\ell}\int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2$ implies that either

$$\max_{[-\ell,\ell]} f < 2\,\frac{C\,\epsilon_2}{4\ell}\int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2\,, \tag{5.50}$$

or

$$\int_{\mathcal{C}_{-2\ell,2\ell}} |u_\theta|^2 = \int_{-2\ell}^{2\ell} f(t)\,dt \ge 2\sqrt{2}\,C\,\epsilon_2\,\frac{\sinh(\ell/\sqrt{2})}{4\ell}\int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2\,. \tag{5.51}$$

The second possibility cannot occur as long as ℓ is sufficiently large so that we have

$$2\sqrt{2}\,C\,\epsilon_2\,\frac{\sinh(\ell/\sqrt{2})}{4\ell} > 1\,. \tag{5.52}$$

Using the upper bound (5.50) for f on $[-\ell,\ell]$ to bound the integral of f gives

$$\int_{\mathcal{C}_{-\ell,\ell}} |u_\theta|^2 \le 2\ell \max_{[-\ell,\ell]} f < C\,\epsilon_2 \int_{\mathcal{C}_{-2\ell,2\ell}} |\nabla u|^2\,. \tag{5.53}$$

The proposition follows by choosing $\epsilon_2 > 0$ so that $C\,\epsilon_2 < \min\{1/4, \delta\}$ and then choosing ℓ so that (5.52) holds. $\square$

4.4. Weak compactness of almost harmonic maps. We will need a compactness theorem for a sequence of maps u^j in $W^{1,2}(\mathbf{S}^2, M)$ which have uniformly bounded energy and are locally well approximated by harmonic maps. Before stating this precisely, it is useful to recall the situation for harmonic maps. Suppose therefore that $u^j : \mathbf{S}^2 \to M$ is a sequence of harmonic maps with $\mathrm{E}(u^j) \le E_0$ for some fixed E_0. After passing to a

subsequence, we can assume that the measures $|\nabla u^j|^2\,dx$ converge and there is a finite set $\mathcal{S}$ of points where the energy concentrates so that:

(5.54) $$\text{If } x \in \mathcal{S}, \text{ then } \inf_{r>0}\left[\lim_{j\to\infty}\int_{B_r(x)}|\nabla u^j|^2\right] \geq \epsilon_{SU}\,.$$

(5.55) $$\text{If } x \notin \mathcal{S}, \text{ then } \inf_{r>0}\left[\lim_{j\to\infty}\int_{B_r(x)}|\nabla u^j|^2\right] < \epsilon_{SU}\,.$$

The constant $\epsilon_{SU} > 0$ comes from Theorem 4.22, so that (5.55) implies uniform $C^{2,\alpha}$ estimates on the u^j's in some neighborhood of x. Hence, Arzela-Ascoli and a diagonal argument give a further subsequence of the u^j's C^2-converging to a harmonic map on every compact subset of $\mathbf{S}^2 \setminus \mathcal{S}$. We will need a more general version of this, where $u^j : \mathbf{S}^2 \to M$ is a sequence of $W^{1,2}$ maps with $\mathrm{E}(u^j) \leq E_0$ that are ϵ_0-almost harmonic in the following sense:

(B_0) If $B \subset \mathbf{S}^2$ is any ball with $\int_B |\nabla u^j|^2 < \epsilon_0$, then there is an energy minimizing map $v : \frac{1}{8}B \to M$ with the same boundary values as u^j on $\partial\frac{1}{8}B$ with
$$\int_{\frac{1}{8}B}\left|\nabla u^j - \nabla v\right|^2 \leq 1/j\,.$$

Lemma 5.16. *Let $\epsilon_0 > 0$ be less than ϵ_{SU}. If $u^j : \mathbf{S}^2 \to M$ is a sequence of $W^{1,2}$ maps satisfying (B_0) and with $E(u^j) \leq E_0$, then there exists a finite collection of points $\{x_1, \dots, x_k\}$, a subsequence still denoted by u^j, and a harmonic map $u : \mathbf{S}^2 \to M$ so that $u^j \to u$ weakly in $W^{1,2}$ and if $K \subset \mathbf{S}^2 \setminus \{x_1, \dots, x_k\}$ is compact, then $u^j \to u$ in $W^{1,2}(K)$. Furthermore, the measures $|\nabla u^j|^2\,dx$ converge to a measure ν with $\epsilon_0 \leq \nu(x_i)$ and $\nu(\mathbf{S}^2) \leq E_0$.*

Proof. After passing to a subsequence, we can assume that:

- The u^j's converge weakly in $W^{1,2}$ to a $W^{1,2}$ map $u : \mathbf{S}^2 \to M$.
- The measures $|\nabla u^j|^2\,dx$ converge to a limiting measure ν with $\nu(\mathbf{S}^2) \leq E_0$.

It follows that there are at most E_0/ϵ_0 points $x_1, \dots, x_k$ with

(5.56) $$\lim_{r\to 0}\nu\left(B_r(x_j)\right) \geq \epsilon_0\,.$$

We will show next that away from the x_i's the convergence is strong in $W^{1,2}$ and u is harmonic. To see this, consider a point $x \notin \{x_1, \dots, x_k\}$. By definition, there exist $r_x > 0$ and J_x so that $\int_{B_{r_x}(x)}|\nabla u^j|^2 < \epsilon_0$ for $j \geq J_x$. In particular, (B_0) applies so we get energy minimizing maps

$v_x^j : \frac{1}{8} B_{r_x}(x) \to M$ that agree with u^j on $\partial \frac{1}{8} B_{r_x}(x)$ and satisfy

$$\int_{\frac{1}{8} B_{r_x}(x)} \left|\nabla v_x^j - \nabla u^j\right|^2 \leq 1/j \,. \tag{5.57}$$

(Here $\frac{1}{8} B_{r_x}(x)$ is the ball in $\mathbf{S}^2$ centered at x so that the stereographic projection Π_x which takes x to $0 \in \mathbb{R}^2$ takes $\frac{1}{8} B_{r_x}(x)$ and $B_{r_x}(x)$ to balls centered at 0 whose radii differ by a factor of 8.) Since $\mathrm{E}(v_x^j) \leq \epsilon_0 \leq \epsilon_{SU}$, it follows from Theorem 4.22 that a subsequence of the v_x^j's converges strongly in $W^{1,2}(\frac{1}{9} B_{r_x}(x))$ to a harmonic map $v_x : \frac{1}{9} B_{r_x}(x) \to M$. Combining this with the triangle inequality and (5.57), we get

$$\begin{aligned} \int_{\frac{1}{9} B_{r_x}(x)} \left|\nabla u^j - \nabla v_x\right|^2 \leq & 2 \int_{\frac{1}{9} B_{r_x}(x)} \left|\nabla u^j - \nabla v_x^j\right|^2 \\ & + 2 \int_{\frac{1}{9} B_{r_x}(x)} \left|\nabla v_x^j - \nabla v_x\right|^2 \to 0 \,. \end{aligned} \tag{5.58}$$

Similarly, this convergence, the triangle inequality, (5.57), and the Dirichlet Poincaré inequality (theorem 3 on page 265 of [**Ev**]; this applies since v_x^j equals u^j on $\partial \frac{1}{8} B_{r_x}(x)$) give

$$\begin{aligned} \int_{\frac{1}{9} B_{r_x}(x)} \left|u^j - v_x\right|^2 \leq & 2 \int_{\frac{1}{8} B_{r_x}(x)} \left|u^j - v_x^j\right|^2 \\ & + 2 \int_{\frac{1}{9} B_{r_x}(x)} \left|v_x^j - v_x\right|^2 \to 0 \,. \end{aligned} \tag{5.59}$$

Combining (5.58) and (5.59), we see that the u^j's converge to v_x strongly in $W^{1,2}(\frac{1}{9} B_{r_x}(x))$. In particular, $u\big|_{\frac{1}{9} B_{r_x}(x)} = v_x$. We conclude that u is harmonic on $\mathbf{S}^2 \setminus \{x_1, \dots, x_k\}$. Furthermore, since any compact $K \subset \mathbf{S}^2 \setminus \{x_1, \dots, x_k\}$ can be covered by a finite number of such ninth-balls, we get that $u^j \to u$ strongly in $W^{1,2}(K)$.

Finally, since u has finite energy, it must have removable singularities at each of the x_i's and, hence, u extends to a harmonic map on all of $\mathbf{S}^2$ (see theorem 3.6 in [**SaUh**]). □

4.5. Almost harmonic maps on cylinders. The main result of this subsection, Proposition 5.18 below, extends Proposition 5.13 from harmonic maps to "almost harmonic" maps. Here "almost harmonic" is made precise in Definition 5.17 below and roughly means that harmonic replacement on certain balls does not reduce the energy by much.

Definition 5.17. *Given $\nu > 0$ and a cylinder $\mathcal{C}_{r_1,r_2}$, we will say that a $W^{1,2}(\mathcal{C}_{r_1,r_2}, M)$ map u is ν-almost harmonic if for any finite collection of disjoint closed balls $\mathcal{B}$ in the conformally equivalent annulus $B_{e^{r_2}} \setminus B_{e^{r_1}} \subset \mathbb{R}^2$*

there is an energy minimizing map $v : \bigcup_{\mathcal{B}} \frac{1}{8}B \to M$ *that equals* u *on* $\bigcup_{\mathcal{B}} \frac{1}{8}\partial B$ *and satisfies*

$$\int_{\bigcup_{\mathcal{B}} \frac{1}{8}B} |\nabla u - \nabla v|^2 \leq \frac{\nu}{2} \int_{\mathcal{C}_{r_1,r_2}} |\nabla u|^2 . \tag{5.60}$$

We have used a slight abuse of notation, since our sets will always be thought of as being subsets of the cylinder; i.e., we identify Euclidean balls in the annulus with their image under the conformal map to the cylinder.

In this subsection and the two that follow it, given $\delta > 0$, the constants $\ell \geq 1$ and $\epsilon_2 > 0$ will be given by Proposition 5.13; these depend only on M and δ.

Proposition 5.18. *Given* $\delta > 0$*, there exists* $\nu > 0$ *(depending on* δ *and* M*) so that if* m *is a positive integer and* u *is* ν*-almost harmonic from* $\mathcal{C}_{-(m+3)\ell,3\ell}$ *to* M *with* $E(u) \leq \epsilon_2$*, then*

$$\int_{\mathcal{C}_{-m\ell,0}} |u_\theta|^2 \leq 7\,\delta \int_{\mathcal{C}_{-(m+3)\ell,3\ell}} |\nabla u|^2 . \tag{5.61}$$

We will prove Proposition 5.18 by using a compactness argument to reduce it to the case of harmonic maps and then appeal to Proposition 5.13. A key difficulty is that there is no upper bound on the length of the cylinder in Proposition 5.18 (i.e., no upper bound on m), so we cannot directly apply the compactness argument. This will be taken care of by dividing the cylinder into subcylinders of a fixed size and then using a covering argument.

4.6. The compactness argument. The next lemma extends Proposition 5.13 from harmonic maps on $\mathcal{C}_{-3\ell,3\ell}$ to almost harmonic maps. The main difference from Proposition 5.18 is that the cylinder is of a fixed size in Lemma 5.19.

Lemma 5.19. *Given* $\delta > 0$*, there exists* $\mu > 0$ *(depending on* δ *and* M*) so that if* u *is a* μ*-almost harmonic map from* $\mathcal{C}_{-3\ell,3\ell}$ *to* M *with* $E(u) \leq \epsilon_2$*, then*

$$\int_{\mathcal{C}_{-\ell,\ell}} |u_\theta|^2 \leq \delta \int_{\mathcal{C}_{-3\ell,3\ell}} |\nabla u|^2 . \tag{5.62}$$

Proof. We will argue by contradiction, so suppose that there exists a sequence u^j of $1/j$-almost harmonic maps from $\mathcal{C}_{-3\ell,3\ell}$ to M with $\mathrm{E}(u^j) \leq \epsilon_2$ and

$$\int_{\mathcal{C}_{-\ell,\ell}} |u^j_\theta|^2 > \delta \int_{\mathcal{C}_{-3\ell,3\ell}} |\nabla u^j|^2 . \tag{5.63}$$

We will show that a subsequence of the u^j's converges to a nonconstant harmonic map that contradicts Proposition 5.13. We will consider two separate cases, depending on whether or not $\mathrm{E}(u^j)$ goes to 0.

Suppose first that $\limsup_{j\to\infty} \mathrm{E}(u^j) > 0$. The upper bound on the energy combined with being $1/j$-almost harmonic (and the compactness of M) allows us to argue as in Lemma 5.16 to get a subsequence that converges in $W^{1,2}$ on compact subsets of $\mathcal{C}_{-3\ell,3\ell}$ to a nonconstant harmonic map $\tilde{u} : \mathcal{C}_{-3\ell,3\ell} \to M$. Furthermore, using the $W^{1,2}$ convergence on $\mathcal{C}_{-\ell,\ell}$ together with the lower semi-continuity of energy, (5.63) implies that

$$\int_{\mathcal{C}_{-\ell,\ell}} |\tilde{u}_\theta|^2 \geq \delta \int_{\mathcal{C}_{-3\ell,3\ell}} |\nabla \tilde{u}|^2 . \tag{5.64}$$

This contradicts Proposition 5.13.

Suppose now that $\mathrm{E}(u^j) \to 0$. Replacing u^j by

$$v^j = \frac{u^j - u^j(0)}{(\mathrm{E}(u^j))^{1/2}} \tag{5.65}$$

gives a sequence of maps to

$$M_j = \frac{M - u^j(0)}{(\mathrm{E}(u^j)))^{1/2}} \tag{5.66}$$

with $\mathrm{E}(v^j) = 1$ and, by (5.63), $\int_{\mathcal{C}_{-\ell,\ell}} |v^j_\theta|^2 > \delta > 0$. Furthermore, the v^j's are also $1/j$-almost harmonic (this property is invariant under dilation), so we can still argue as in Lemma 5.16 to get a subsequence that converges in $W^{1,2}$ on compact subsets of $\mathcal{C}_{-3\ell,3\ell}$ to a harmonic map $v : \mathbf{S}^2 \to \mathbb{R}^N$ (here we are using that a subsequence of the M_j's converges to an affine space). As before, (5.63) implies that

$$\int_{\mathcal{C}_{-\ell,\ell}} |v_\theta|^2 \geq \delta \int_{\mathcal{C}_{-3\ell,3\ell}} |\nabla v|^2 . \tag{5.67}$$

This time our normalization gives $\int_{\mathcal{C}_{-\ell,\ell}} |v_\theta|^2 \geq \delta$ so that v contradicts Proposition 5.13, completing the proof. □

4.7. The proof of Proposition 5.18.

Proof of Proposition 5.18. For each integer $j = 0, \ldots, m$, let $\mathcal{C}(j) = \mathcal{C}_{-(j+3)\ell,(3-j)\ell}$ and let $\mu > 0$ be given by Lemma 5.19. We will say that the j-th cylinder $\mathcal{C}(j)$ is *good* if the restriction of u to $\mathcal{C}(j)$ is μ-almost harmonic; otherwise, we will say that $\mathcal{C}(j)$ is *bad*.

On each good $\mathcal{C}(j)$, we apply Lemma 5.19 to get

$$\int_{\mathcal{C}_{-(j+1)\ell,(1-j)\ell}} |u_\theta|^2 \leq \delta \int_{\mathcal{C}(j)} |\nabla u|^2 , \tag{5.68}$$

so that summing this over the good j's gives

(5.69)
$$\sum_{j\,\mathrm{good}} \int_{\mathcal{C}_{-(j+1)\ell,(1-j)\ell}} |u_\theta|^2 \leq \delta \sum_{j\,\mathrm{good}} \int_{\mathcal{C}(j)} |\nabla u|^2 \leq 6\,\delta \int_{\mathcal{C}_{-(m+3)\ell,3\ell}} |\nabla u|^2 \,,$$

where the last inequality used that each $\mathcal{C}_{i,i+1}$ is contained in at most 6 of the $\mathcal{C}(j)$'s.

We will complete the proof by showing that the total energy (not just the θ-energy) on the bad $\mathcal{C}(j)$'s is small. By definition, for each bad $\mathcal{C}(j)$, we can choose a finite collection of disjoint closed balls $\mathcal{B}_j$ in $\mathcal{C}(j)$ so that if $v : \frac{1}{8}\mathcal{B}_j \to M$ is any energy-minimizing map that equals u on $\partial \frac{1}{8}\mathcal{B}_j$, then

$$\int_{\frac{1}{8}\mathcal{B}_j} |\nabla u - \nabla v|^2 \geq a_j > \mu \int_{\mathcal{C}(j)} |\nabla u|^2 \,. \tag{5.70}$$

Since the interior of each $\mathcal{C}(j)$ intersects only the $\mathcal{C}(k)$'s with $0 < |j-k| \leq 5$, we can divide the bad $\mathcal{C}(j)$'s into ten subcollections so that the interiors of the $\mathcal{C}(j)$'s in each subcollection are pairwise disjoint. In particular, one of these disjoint subcollections, call it Γ, satisfies

$$\sum_{j \in \Gamma} a_j \geq \frac{1}{10} \sum_{j\,\mathrm{bad}} a_j \geq \frac{1}{10} \sum_{j\,\mathrm{bad}} \mu \int_{\mathcal{C}(j)} |\nabla u|^2 \,, \tag{5.71}$$

where the last inequality used (5.70).

However, since $\bigcup_{j\in\Gamma} \mathcal{B}_j$ is itself a finite collection of disjoint closed balls in the entire cylinder $\mathcal{C}_{-(m+3)\ell,3\ell}$ and u is ν-almost harmonic on $\mathcal{C}_{-(m+3)\ell,3\ell}$, we get that

$$\frac{\mu}{10} \sum_{j\,\mathrm{bad}} \int_{\mathcal{C}(j)} |\nabla u|^2 \leq \nu \int_{\mathcal{C}_{-(m+3)\ell,3\ell}} |\nabla u|^2 \,. \tag{5.72}$$

To get the proposition, combine (5.69) with (5.72) to get

$$\int_{\mathcal{C}_{-m\ell,0}} |u_\theta|^2 \leq \left(6\,\delta + \frac{10\,\nu}{\mu} \right) \int_{\mathcal{C}_{-(m+3)\ell,3\ell}} |\nabla u|^2 \,. \tag{5.73}$$

Finally, choosing ν sufficiently small completes the proof. □

4.8. Bubble compactness. We will now prove Proposition 5.12 using a variation of the renormalization procedure developed in [**PkW**] for pseudo-holomorphic curves and later used in [**Pk**] for harmonic maps. A key point in the proof will be that the uniform energy bound, (A), and (B) are all dilation invariant, so they apply also to the compositions of the u^j's with any sequence of conformal dilations of $\mathbf{S}^2$.

Proof of Proposition 5.12. We will use the energy bound and (B) to show that a subsequence of the u^j's converges in the sense of (B1), (B2),

and (B3) of Definition 3.26 to a collection of harmonic maps. We will then come back and use (A) and (B) to show that the energy equality (B4) also holds. Hence, the subsequence bubble converges and, thus by Proposition 3.27, also varifold converges.

Set $\delta = 1/21$ and let $\ell \geq 1$ and $\epsilon_2 > 0$ be given by Proposition 5.13. Set $\epsilon_3 = \min\{\epsilon_0/2, \epsilon_2\}$.

Step 1: Initial compactness. Lemma 5.16 gives a finite collection of singular points $\mathcal{S}_0 \subset \mathbf{S}^2$, a harmonic map $v_0 : \mathbf{S}^2 \to M$, and a subsequence (still denoted u^j) that converges to v_0 weakly in $W^{1,2}(\mathbf{S}^2)$ and strongly in $W^{1,2}(K)$ for any compact subset $K \subset \mathbf{S}^2 \setminus \mathcal{S}_0$. Furthermore, the measures $|\nabla u^j|^2\,dx$ converge to a measure ν_0 with $\nu_0(\mathbf{S}^2) \leq E_0$ and each singular point in $x \in \mathcal{S}_0$ has $\nu_0(x) \geq \epsilon_0$.

Step 2: Renormalizing at a singular point. Suppose that $x \in \mathcal{S}_0$ is a singular point from the first step. Fix a radius $\rho > 0$ so that x is the only singular point in $B_{2\rho}(x)$ and $\int_{B_\rho(x)} |\nabla v_0|^2 < \epsilon_3/3$. For each j, let $r_j > 0$ be the smallest radius so that

$$\inf_{y \in B_{\rho - r_j}(x)} \int_{B_\rho(x) \setminus B_{r_j}(y)} |\nabla u^j|^2 = \epsilon_3\,, \tag{5.74}$$

and choose a ball $B_{r_j}(y_j) \subset B_\rho(x)$ with

$$\int_{B_\rho(x) \setminus B_{r_j}(y_j)} |\nabla u^j|^2 = \epsilon_3\,. \tag{5.75}$$

Since the u^j's converge to v_0 on compact subsets of $B_\rho(x) \setminus \{x\}$, we get that $y_j \to x$ and $r_j \to 0$. For each j, let

$$\Psi_j : \mathbb{R}^2 \to \mathbb{R}^2 \tag{5.76}$$

be the "dilation" that takes $B_{r_j}(y_j)$ to the unit ball $B_1(0) \subset \mathbb{R}^2$. By dilation invariance, the dilated maps $\tilde{u}_1^j = u^j \circ \Psi_j^{-1}$ still satisfy (B) and have the same energy. Hence, Lemma 5.16 gives a subsequence (still denoted by $\tilde{u}_1^j$), a finite singular set $\mathcal{S}_1$, and a harmonic map v_1 so that the $\tilde{u}_1^j \circ \Pi$'s converge to v_1 weakly in $W^{1,2}(\mathbf{S}^2)$ and strongly in $W^{1,2}(K)$ for any compact subset $K \subset \mathbf{S}^2 \setminus \mathcal{S}_1$. Moreover, the measures $|\nabla \tilde{u}_1^j \circ \Pi|^2\,dx$ converge to a measure ν_1.

The choice of the balls $B_{r_j}(y_j)$ guarantees that $\nu_1(\mathbf{S}^2 \setminus \{p^+\}) \leq \nu_0(x)$ and $\nu_1(S^-) \leq \nu_0(x) - \epsilon_3$. (Recall that stereographic projection Π takes the open southern hemisphere S^- to the open unit ball in $\mathbb{R}^2$.) The key point for iterating this is the following claim:

($\star$) The maximal energy concentration at any $y \in \mathcal{S}_1 \setminus \{p^+\}$ is at most $\nu_0(x) - \epsilon_3/3$.

Since the energy at a singular point or the energy for a nontrivial harmonic map is at least $\epsilon_0 > \epsilon_3$, the only way that $(\star)$ could possibly fail is if v_1 is constant, $\mathcal{S}_1$ is exactly two points p^+ and y, and at most $\epsilon_3/3$ of $\nu_0(x)$ escapes at p^+. However, this would imply that all but at most $2\epsilon_3/3$ of the $\int_{B_\rho(x)} |\nabla u^j|^2$ is in $B_{t_j}(y_j)$ with $\frac{t_j}{r_j} \to 0$ which contradicts the minimality of r_j.

Step 3: Repeating this. We repeat this blowing up construction at the remaining singular points in $\mathcal{S}_0$, as well as each of the singular points $\mathcal{S}_1$ in the southern hemisphere, etc., to get new limiting harmonic maps and new singular points to blow up at. It follows from $(\star)$ that this must terminate after at most $3\,E_0/\epsilon_3$ steps.

Step 4: The necks. We have shown that the u^j's converge to a collection of harmonic maps in the sense of (B1), (B2), and (B3). It remains to show (B4), i.e., that the v_k's accounted for all of the energy in the sequence u^j and no energy was lost in the limit.

To understand how energy could be lost, it is useful to re-examine what happens to the energy during the blow up process. At each stage in the blow up process, energy is "taken from" a singular point x and then goes to one of two places:

- It can show up in the new limiting harmonic map of a singular point in $\mathbf{S}^2 \setminus \{p^+\}$.
- It can disappear at the north pole p^+ (i.e., $\nu_1(\mathbf{S}^2 \setminus \{p^+\}) < \nu_0(x)$).

In the first case, the energy is accounted for in the limit or survives to a later stage. However, in the second case, the energy is lost for good, so this is what we must rule out.

We will argue by contradiction, so suppose that

$$\nu_1(\mathbf{S}^2 \setminus \{p^+\}) < \nu_0(x) - \hat{\delta} \tag{5.77}$$

for some $\hat{\delta} > 0$. (Note that we must have $\hat{\delta} \leq \epsilon_3$.) Using the notation in step 1, suppose therefore that $A_j = B_{s_j}(y_j) \setminus B_{t_j}(y_j)$ are annuli with

$$s_j \to 0\,, \ \frac{t_j}{r_j} \to \infty\,, \text{ and } \int_{A_j} |\nabla u_j|^2 \geq \hat{\delta} > 0\,. \tag{5.78}$$

There is obviously quite a bit of freedom in choosing s_j and t_j. In particular, we can choose a sequence $\lambda_j \to \infty$ so that the annuli

$$\tilde{A}_j = B_{\rho/2}(y_j) \setminus B_{t_j/\lambda_j}(y_j) \tag{5.79}$$

also satisfy this, i.e., $\lambda_j\, s_j \to 0$ and $t_j/(\lambda_j r_j) \to \infty$. It follows from (5.78) and the definition of the r_j's that

$$\int_{\tilde{A}_j} |\nabla u^j|^2 \leq \epsilon_3 \leq \epsilon_2 \,. \tag{5.80}$$

However, combining this with Proposition 5.18 (with $\delta = 1/21$) shows that the area must be strictly less than the energy for j large, contradicting (A), and thus completing the proof. □

5. Minimal Spheres and the Width

We will next define a two-dimensional width, analogous to the one-dimensional width defined earlier, and use it to produce minimal spheres.

5.1. Width. We will now define a two-dimensional version of the width defined earlier for sweepouts by curves. We sometimes call this the 2-width, but, when there is no possible confusion, then we will just call it the width. Let Ω be the set of continuous maps

$$\sigma : \mathbf{S}^2 \times [0,1] \to M$$

so that:

- For each $t \in [0,1]$ the map $\sigma(\cdot, t)$ is in $C^0 \cap W^{1,2}$.
- The map $t \to \sigma(\cdot, t)$ is continuous from $[0,1]$ to $C^0 \cap W^{1,2}$.
- σ maps $\mathbf{S}^2 \times \{0\}$ and $\mathbf{S}^2 \times \{1\}$ to points.

Given a map $\beta \in \Omega$, the homotopy class Ω_β is defined to be the set of maps $\sigma \in \Omega$ that are homotopic to β through maps in Ω. We will call any such β a *sweepout*; some authors use a more restrictive notion where β must also induce a degree one map from $\mathbf{S}^3$ to M. We will, in fact, be most interested in the case where β induces a map from $\mathbf{S}^3$ to M in a nontrivial class[7] in $\pi_3(M)$. The reason for this is that the width is positive in this case and, as we will see, equal to the area of a nonempty collection of minimal 2-spheres.

The (energy) width $W_E = W_E(\beta, M)$ associated to the homotopy class Ω_β is defined by taking the infimum of the maximum of the energy of each slice. That is, set

$$W_E = \inf_{\sigma \in \Omega_\beta} \max_{t \in [0,1]} \mathrm{E}\left(\sigma(\cdot, t)\right), \tag{5.81}$$

where the energy is given by

$$\mathrm{E}\left(\sigma(\cdot, t)\right) = \frac{1}{2} \int_{\mathbf{S}^2} \left|\nabla_x \sigma(x,t)\right|^2 \, dx \,. \tag{5.82}$$

[7]For example, when M is a homotopy 3-sphere and the induced map has degree one.

Even though this type of construction is always called min-max, it is really inf-max. That is, for each (smooth) sweepout one looks at the maximal energy of the slices and then takes the infimum over all sweepouts in a given homotopy class. The width is always nonnegative by definition, and positive when the homotopy class of β is nontrivial. Positivity can, for instance, be seen directly using [**Jo**]. Namely, page 125 in [**Jo**] shows that if $\max_t \mathrm{E}(\sigma(\cdot,t))$ is sufficiently small (depending on M), then σ is homotopically trivial.[8]

One could alternatively define the width using area rather than energy by setting

$$W_A = \inf_{\sigma\in\Omega_\beta} \max_{t\in[0,1]} \mathrm{Area}\,(\sigma(\cdot,t))\,. \tag{5.83}$$

The area of a $W^{1,2}$ map $u : \mathbf{S}^2 \to \mathbb{R}^N$ is by definition the integral of the Jacobian

$$J_u = \sqrt{\det\left(du^T\, du\right)}\,, \tag{5.84}$$

where du is the differential of u and du^T is its transpose. That is, if e_1, e_2 is an orthonormal frame on $D \subset \mathbf{S}^2$, then

$$J_u = \left(|u_{e_1}|^2\,|u_{e_2}|^2 - \langle u_{e_1}, u_{e_2}\rangle^2\right)^{\frac{1}{2}} \le \frac{1}{2}\,|du|^2 \tag{5.85}$$

and

$$\mathrm{Area}(u\big|_D) = \int_D J_u \le \mathrm{E}(u\big|_D)\,. \tag{5.86}$$

Consequently, area is less than or equal to energy with equality if and only if $\langle u_{e_1}, u_{e_2}\rangle$ and $|u_{e_1}|^2 - |u_{e_2}|^2$ are zero (as L^1 functions). In the case of equality, we say that u is *almost conformal*; see (5.86). As in the classical Plateau problem (see Theorem 4.1 in Chapter 4), energy is somewhat easier to work with in proving the existence of minimal surfaces. The next proposition shows that $W_E = W_A$ as for the Plateau problem (clearly, $W_A \le W_E$ by the discussion above). Therefore, we will drop the subscript and just write W.

Proposition 5.20 (Colding-Minicozzi, [**CM27**]). *$W_E = W_A$.*

By (5.86), Proposition 5.20 follows once we show that $W_E \le W_A$. The corresponding result for the Plateau problem is proven by taking a minimizing sequence for area and reparametrizing to make these maps conformal, i.e., choosing isothermal coordinates. There are a few technical difficulties in carrying this out since the pullback metric may be degenerate and is only in L^1, while the existence of isothermal coordinates requires that the induced metric be positive and bounded; see, e.g., proposition 5.4 in [**ScW**].

[8] Corollary 5.10 can be used to give a different proof.

We will follow the same approach here, the difference is that we need the reparametrizations to vary continuously with t.

5.2. Density of smooth mappings. The next lemma observes that the regularization using convolution of Schoen and Uhlenbeck in the proposition in section 4 of [**ScU2**] is continuous.

Lemma 5.21. *Given $\gamma \in \Omega$ and $\epsilon > 0$, there exists a regularization $\tilde{\gamma} \in \Omega_\gamma$ so that*

$$\max_t \; ||\tilde{\gamma}(\cdot, t) - \gamma(\cdot, t)||_{W^{1,2}} \leq \epsilon \,, \tag{5.87}$$

each slice $\tilde{\gamma}(\cdot, t)$ is C^2, and the map $t \to \tilde{\gamma}(\cdot, t)$ is continuous from $[0, 1]$ to $C^2(\mathbf{S}^2, M)$.

Proof. Since M is smooth, compact and embedded, there exists a $\delta > 0$ so that for each x in the δ-tubular neighborhood M_δ of M in $\mathbb{R}^N$, there is a unique closest point $\Pi(x) \in M$ and so the map $x \to \Pi(x)$ is smooth. Π is called *nearest point projection.*

Given y in the open ball $B_1(0) \subset \mathbb{R}^3$, define $T_y : \mathbf{S}^2 \to \mathbf{S}^2$ by $T_y(x) = \frac{x-y}{|x-y|}$. Since each T_y is smooth and these maps depend smoothly on y, it follows that the map $(y, f) \to f \circ T_y$ is continuous from

$$B_1(0) \times C^0 \cap W^{1,2}(\mathbf{S}^2, \mathbb{R}^N) \to C^0 \cap W^{1,2}(\mathbf{S}^2, \mathbb{R}^N) \tag{5.88}$$

(this is clear for $f \in C^1$ and follows for $C^0 \cap W^{1,2}$ by density). Therefore, since T_0 is the identity, given $f \in C^0 \cap W^{1,2}(\mathbf{S}^2, \mathbb{R}^N)$ and $\mu > 0$, there exists $r > 0$ so that

$$\sup_{|y| \leq r} \; ||f \circ T_y - f||_{C^0 \cap W^{1,2}} < \mu \,. \tag{5.89}$$

Applying this to $\gamma(\cdot, t)$ for each t and using that $t \to \gamma(\cdot, t)$ is continuous to $C^0 \cap W^{1,2}$ and $[0, 1]$ is compact, we get $\bar{r} > 0$ with

$$\sup_{t \in [0,1]} \; \sup_{|y| \leq \bar{r}} \; ||T_y \gamma(\cdot, t) - \gamma(\cdot, t)||_{C^0 \cap W^{1,2}} < \mu \,. \tag{5.90}$$

Next fix a smooth radial mollifier $\phi \geq 0$ with integral one and compact support in the unit ball in $\mathbb{R}^3$. For each $r \in (0, 1)$, define $\phi_r(x) = r^{-3} \, \phi(x/r)$ and set

$$\gamma_r(x, t) = \int_{B_r(0)} \phi_r(y) \gamma(T_y(x) \, , t) \, dy = \int_{B_r(x)} \phi_r(x - y) \gamma(\frac{y}{|y|} \, , t) \, dy \,. \tag{5.91}$$

We have the following standard properties of convolution with a mollifier (see, e.g., section 5.3 and appendix C.4 in [**Ev**]): First, each $\gamma_r(\cdot, t)$ is smooth

and for each k the map $t \to \gamma_r(\cdot, t)$ is continuous from $[0,1]$ to $C^k(\mathbf{S}^2, \mathbb{R}^N)$. Second,

$$(5.92) \qquad ||\gamma_r(\cdot,t) - \gamma(\cdot,t)||^2_{C^0} \leq \sup_{|y|\leq r} ||T_y\gamma(\cdot,t) - \gamma(\cdot,t)||^2_{C^0} ,$$
$$||\nabla\gamma_r(\cdot,t) - \nabla\gamma(\cdot,t)||^2_{L^2} \leq \sup_{|y|\leq r} ||T_y\gamma(\cdot,t) - \gamma(\cdot,t)||^2_{L^2} .$$

It follows from (5.92) and (5.90) that for $r \leq \bar{r}$ and all t we have

$$(5.93) \qquad ||\gamma_r(\cdot,t) - \gamma(\cdot,t)||_{C^0\cap W^{1,2}} < \mu .$$

The map $\gamma_r(\cdot,t)$ may not land in M, but it is in M_δ when μ is small by (5.93). Hence, the map $\tilde{\gamma}(x,t) = \Pi \circ \gamma_r(x,t)$ satisfies (5.87), each slice $\tilde{\gamma}(\cdot,t)$ is C^2, and $t \to \tilde{\gamma}(\cdot,t)$ is continuous from $[0,1]$ to $C^2(\mathbf{S}^2, M)$. Finally, $s \to \tilde{\gamma}_{sr}$ is an explicit homotopy connecting $\tilde{\gamma}$ and γ. □

5.3. Equivalence of energy and area. We will also need the existence of isothermal coordinates, taking special care on the dependence on the metric. Let $\mathbf{S}^2_{g_0}$ denote the round metric on $\mathbf{S}^2$ with constant curvature one.

Lemma 5.22. *Given a C^1 metric $\tilde{g}$ on $\mathbf{S}^2$, there is a unique orientation preserving $C^{1,1/2}$ conformal diffeomorphism $h_{\tilde{g}} : \mathbf{S}^2_{g_0} \to \mathbf{S}^2_{\tilde{g}}$ that fixes 3 given points.*

Moreover, if $\tilde{g}_1$ and $\tilde{g}_2$ are two C^1 metrics that are both $\geq \epsilon\, g_0$ for some $\epsilon > 0$, then

$$(5.94) \qquad ||h_{\tilde{g}_1} - h_{\tilde{g}_2}||_{C^0\cap W^{1,2}} \leq C\, ||\tilde{g}_1 - \tilde{g}_2||_{C^0} ,$$

where the constant C depends on ϵ and the maximum of the C^1 norms of the $\tilde{g}_i$'s.

Proof. The Riemann mapping theorem for variable metrics (see theorem 3.1.1 and corollary 3.1.1 in [**Jo**]; cf. [**ABe**] or [**Mo2**]) gives the conformal diffeomorphism $h_{\tilde{g}} : \mathbf{S}^2_{g_0} \to \mathbf{S}^2_{\tilde{g}}$.

We will separately bound the C^0 and $W^{1,2}$ norms. First, lemma 17 in [**ABe**] gives

$$(5.95) \qquad ||h_{\tilde{g}_1} - h_{\tilde{g}_2}||_{C^0} \leq C_1\, ||\tilde{g}_1 - \tilde{g}_2||_{C^0} ,$$

where C_1 depends on ϵ and the C^0 norms of the metrics. Second, theorem 8 in [**ABe**] gives a uniform L^p bound for $\nabla(h_{\tilde{g}_1} - h_{\tilde{g}_2})$ on any unit ball in $\mathbf{S}^2$ where $p > 2$ by (8) in [**ABe**]

$$(5.96) \qquad ||\nabla(h_{\tilde{g}_1} - h_{\tilde{g}_2})||_{L^p(B_1)} \leq C_2\, ||\tilde{g}_1 - \tilde{g}_2||_{C^0(\mathbf{S}^2)} ,$$

where C_2 depends on ϵ and the C^0 norms of the metrics. Covering $\mathbf{S}^2$ by a finite collection of unit balls and applying Hölder's inequality gives the desired energy bound. □

We can now prove the equivalence of the two widths.

Proof of Proposition 5.20. By (5.86), we have that $W_A \leq W_E$. To prove that $W_E \leq W_A$, given $\epsilon > 0$, let $\gamma \in \Omega_\beta$ be a sweepout with

$$\max_{t \in [0,1]} \text{Area}\,(\gamma(\cdot, t)) < W_A + \epsilon/2\,. \tag{5.97}$$

By Lemma 5.21, there is a regularization $\tilde{\gamma} \in \Omega_\beta$ so that each slice $\tilde{\gamma}(\cdot, t)$ is C^2, the map $t \to \tilde{\gamma}(\cdot, t)$ is continuous from $[0, 1]$ to $C^2(\mathbf{S}^2, M)$, and (also by (3.112))

$$\max_t \text{Area}\,(\tilde{\gamma}(\cdot, t)) < W_A + \epsilon\,. \tag{5.98}$$

The maps $\tilde{\gamma}(\cdot, t)$ induce a continuous one-parameter family of pullback (possibly degenerate) C^1 metrics $g(t)$ on $\mathbf{S}^2$. Lemma 5.22 requires that the metrics be nondegenerate, so define perturbed metrics $\tilde{g}(t) = g(t) + \delta\, g_0$. For each t, Lemma 5.22 gives $C^{1,1/2}$ conformal diffeomorphisms $h_t : \mathbf{S}^2_{g_0} \to \mathbf{S}^2_{\tilde{g}(t)}$ that vary continuously in $C^0 \cap W^{1,2}(\mathbf{S}^2, \mathbf{S}^2)$. The continuity of $t \to \tilde{\gamma}(\cdot, t) \circ h_t$ as a map from $[0, 1]$ to $C^0 \cap W^{1,2}(\mathbf{S}^2, M)$ follows from this, the continuity of $t \to \tilde{\gamma}(\cdot, t)$ in C^2, and the chain rule.

We will now use the conformality of the map h_t to control the energy of the composition as in proposition 5.4 of [**ScW**]. Namely, we have that

$$\begin{aligned}
&\mathrm{E}\,(\tilde{\gamma}(\cdot, t) \circ h_t) = \mathrm{E}\,(h_t : \mathbf{S}^2_{g_0} \to \mathbf{S}^2_{g(t)}) \leq \mathrm{E}\,(h_t : \mathbf{S}^2_{g_0} \to \mathbf{S}^2_{\tilde{g}(t)}) \\
&= \text{Area}\,(\mathbf{S}^2_{\tilde{g}(t)}) = \int_{\mathbf{S}^2} [\det(g_0^{-1}\, g(t)) + \delta\ \mathrm{Tr}(g_0^{-1}\, g(t)) + \delta^2]^{1/2}\, dvol_{g_0} \\
&\leq \text{Area}\,(\mathbf{S}^2_{g(t)}) + 4\pi\, [\delta^2 + 2\,\delta \sup_t |g_0^{-1}\, g(t)|]^{1/2}\,.
\end{aligned} \tag{5.99}$$

Choose $\delta > 0$ so that $4\pi\, [\delta^2 + 2\,\delta\, \sup_t |g_0^{-1}\, g(t)|]^{1/2} < \epsilon$.

We would be done if $\tilde{\gamma}(\cdot, t) \circ h_t$ was homotopic to $\tilde{\gamma}$. However, the space of orientation preserving diffeomorphisms of $\mathbf{S}^2$ is homotopic to $\mathbb{RP}^3$ by Smale's theorem. To get around this, note that $t \to ||\tilde{\gamma}(\cdot, t)||_{C^2}$ is continuous and zero when $t = 1$, thus for some $\tau < 1$,

$$\sup_{t \geq \tau} ||\tilde{\gamma}(\cdot, t)||_{C^2} \leq \frac{\epsilon}{\sup_{t \in [0,1]} ||h_t||^2_{W^{1,2}}}\,. \tag{5.100}$$

Consequently, if we set $\tilde{h}_t$ equal to $h_t \equiv h(t)$ on $[0, \tau]$ and equal to $h(\tau(1 - t)/(1 - \tau))$ on $[\tau, 1]$, then (5.99) and (5.100) imply that

$$\max_{t \in [0,1]} \mathrm{E}\,(\tilde{\gamma}(\cdot, t) \circ \tilde{h}_t) \leq W_A + 2\,\epsilon\,. \tag{5.101}$$

Moreover, the map $\tilde{\gamma}(\cdot, t) \circ \tilde{h}_t$ is also in Ω. Finally, replacing τ by $s\tau$ and taking $s \to 0$ gives an explicit homotopy in Ω from $\tilde{\gamma}(\cdot, t) \circ \tilde{h}_t$ to $\tilde{\gamma}(\cdot, t)$. $\square$

5.4. The energy decreasing map on sweepouts. Now that we have developed the basic properties of harmonic replacement, we will define an energy decreasing map from Ω to itself that preserves the homotopy class (i.e., maps each Ω_β to itself) and record its key properties. This should be thought of as a generalization of Birkhoff's curve shortening process that plays a similar role when tightening a sweepout by curves.

Theorem 5.23 (Colding-Minicozzi, [**CM27**]). *There is a constant $\epsilon_0 > 0$ and a continuous function $\Psi : [0, \infty) \to [0, \infty)$ with $\Psi(0) = 0$, both depending on M, so that given any $\tilde{\gamma} \in \Omega$ without nonconstant harmonic slices and $W > 0$, there exists $\gamma \in \Omega_{\tilde{\gamma}}$ so that $E(\gamma(\cdot, t)) \leq E(\tilde{\gamma}(\cdot, t))$ for each t and so for each t with $E(\tilde{\gamma}(\cdot, t)) \geq W/2$:*

(B_Ψ) *If $\mathcal{B}$ is any finite collection of disjoint closed balls in $\mathbf{S}^2$ with $\int_{\bigcup_{\mathcal{B}} B} |\nabla \gamma(\cdot, t)|^2 < \epsilon_0$ and $v : \bigcup_{\mathcal{B}} \frac{1}{8} B \to M$ is an energy minimizing map equal to $\gamma(\cdot, t)$ on $\bigcup_{\mathcal{B}} \frac{1}{8} \partial B$, then*

$$\int_{\bigcup_{\mathcal{B}} \frac{1}{8} B} |\nabla \gamma(\cdot, t) - \nabla v|^2 \leq \Psi \left[E(\tilde{\gamma}(\cdot, t)) - E(\gamma(\cdot, t)) \right] .$$

We will construct $\gamma(\cdot, t)$ from $\tilde{\gamma}(\cdot, t)$ by harmonic replacement on a family of balls in $\mathbf{S}^2$ varying continuously in t. The balls will be chosen in Lemma 5.25 below. Throughout this subsection, $\epsilon_1 > 0$ will be the small energy constant (depending on M) given by Theorem 5.8.

Given $\sigma \in \Omega$ and $\epsilon \in (0, \epsilon_1]$, define the maximal improvement from harmonic replacement on families of balls with energy at most ϵ by

$$e_{\sigma,\epsilon}(t) = \sup_{\mathcal{B}} \{ \mathrm{E}(\sigma(\cdot, t)) - \mathrm{E}(H(\sigma(\cdot, t), \frac{1}{2}\mathcal{B})) \} , \tag{5.102}$$

where the supremum is over all finite collections $\mathcal{B}$ of disjoint closed balls where the total energy of $\sigma(\cdot, t)$ on $\mathcal{B}$ is at most ϵ. Observe that $e_{\sigma,\epsilon}(t)$ is nonnegative, monotone nondecreasing in ϵ, and is positive if $\sigma(\cdot, t)$ is not harmonic.

Lemma 5.24. *If $\sigma(\cdot, t)$ is not harmonic and $\epsilon \in (0, \epsilon_1]$, then there is an open interval I^t containing t so that $e_{\sigma,\epsilon/2}(s) \leq 2\, e_{\sigma,\epsilon}(t)$ for all s in the double interval $2I^t$.*

Proof. By (5.35) in Corollary 5.10, there exists $\delta_1 > 0$ (depending on t) so that if

$$||\sigma(\cdot, t) - \sigma(\cdot, s)||_{C^0 \cap W^{1,2}} < \delta_1 \tag{5.103}$$

and $\mathcal{B}$ is a finite collection of disjoint closed balls where both $\sigma(\cdot,t)$ and $\sigma(\cdot,s)$ have energy at most ϵ_1, then

$$\left|\mathrm{E}(H(\sigma(\cdot,s),\frac{1}{2}\mathcal{B}))-\mathrm{E}(H(\sigma(\cdot,t),\frac{1}{2}\mathcal{B}))\right|\leq e_{\sigma,\epsilon}(t)/2\,. \tag{5.104}$$

Here we have used that $e_{\sigma,\epsilon}(t)>0$ since $\sigma(\cdot,t)$ is not harmonic. Since $t\to\sigma(\cdot,t)$ is continuous as a map to $C^0\cap W^{1,2}$, we can choose I^t so that for all $s\in 2\,I^t$ (5.103) holds and

$$\frac{1}{2}\int_{\mathbf{S}^2}\left||\nabla\sigma(\cdot,t)|^2-|\nabla\sigma(\cdot,s)|^2\right|\leq\min\left\{\frac{\epsilon}{2},\frac{e_{\sigma,\epsilon}(t)}{2}\right\}. \tag{5.105}$$

Suppose now that $s\in 2I^t$ and the energy of $\sigma(\cdot,s)$ is at most $\epsilon/2$ on a collection $\mathcal{B}$. It follows from (5.105) that the energy of $\sigma(\cdot,t)$ is at most ϵ on $\mathcal{B}$. Combining (5.104) and (5.105) gives

(5.106)
$$\left|\mathrm{E}(\sigma(\cdot,s))-\mathrm{E}(H(\sigma(\cdot,s),\frac{1}{2}\mathcal{B}))-\mathrm{E}(\sigma(\cdot,t))+\mathrm{E}(H(\sigma(\cdot,t),\frac{1}{2}\mathcal{B}))\right|\leq e_{\sigma,\epsilon}(t)\,.$$

Since this applies to any such $\mathcal{B}$, we get that $e_{\sigma,\epsilon/2}(s)\leq 2\,e_{\sigma,\epsilon}(t)$. □

Given a sweepout with no harmonic slices, the next lemma constructs finitely many collections of balls so that harmonic replacement on at least one of these collections strictly decreases the energy. In addition, each collection consists of finitely many pairwise disjoint closed balls.

Lemma 5.25. *If $W>0$ and $\tilde{\gamma}\in\Omega$ has no non-constant harmonic slices, then we get an integer m (depending on $\tilde{\gamma}$), m collections of balls $\mathcal{B}_1,\dots,\mathcal{B}_m$ in $\mathbf{S}^2$ where the balls in each collection $\mathcal{B}_j$ are pairwise disjoint, and m continuous functions $r_1,\dots,r_m:[0,1]\to[0,1]$ so that for each t:*

1. *At most two $r_j(t)$'s are positive and $\sum_{B\in\mathcal{B}_j}\frac{1}{2}\int_{r_j(t)B}|\nabla\tilde{\gamma}(\cdot,t)|^2<\epsilon_1/3$ for each j.*
2. *If $E(\tilde{\gamma}(\cdot,t))\geq W/2$, then there exists $j(t)$ so that harmonic replacement on $\frac{r_{j(t)}}{2}\mathcal{B}_{j(t)}$ decreases energy by at least $e_{\tilde{\gamma},\epsilon_1/8}(t)/8$.*

Proof. Since the energy of the slices is continuous in t, the set

$$I=\{t\,|\,\mathrm{E}(\tilde{\gamma}(\cdot,t))\geq W/2\} \tag{5.107}$$

is compact. For each $t\in I$, choose a finite collection $\mathcal{B}^t$ of disjoint closed balls in $\mathbf{S}^2$ with $\frac{1}{2}\int_{\bigcup_{\mathcal{B}^t}}|\nabla\tilde{\gamma}(\cdot,t)|^2\leq\epsilon_1/4$ so

$$\mathrm{E}(\gamma(\cdot,t))-\mathrm{E}(H(\gamma(\cdot,t),\frac{1}{2}\,\mathcal{B}^t))\geq\frac{e_{\tilde{\gamma},\epsilon_1/4}(t)}{2}>0\,. \tag{5.108}$$

Lemma 5.24 gives an open interval I^t containing t so that for all $s\in 2I^t$

$$e_{\tilde{\gamma},\epsilon_1/8}(s)\leq 2\,e_{\tilde{\gamma},\epsilon_1/4}(t)\,. \tag{5.109}$$

Using the continuity of $\tilde{\gamma}(\cdot, s)$ in $C^0 \cap W^{1,2}$ and Corollary 5.10, we can shrink I^t so that $\tilde{\gamma}(\cdot, s)$ has energy at most $\epsilon_1/3$ in $\mathcal{B}^t$ for $s \in 2I^t$ and, in addition,

$$\left| \mathrm{E}(\gamma(\cdot, s)) - \mathrm{E}(H(\gamma(\cdot, s), \frac{1}{2}\mathcal{B}^t)) - \mathrm{E}(\gamma(\cdot, t)) + \mathrm{E}(H(\gamma(\cdot, t), \frac{1}{2}\mathcal{B}^t)) \right|$$

$$(5.110) \qquad \leq \frac{e_{\tilde{\gamma}, \epsilon_1/4}(t)}{4} .$$

Since I is compact, we can cover I by finitely many I^t's, say $I^{t_1}, \dots, I^{t_m}$. Moreover, after discarding some of the intervals, we can arrange that:

> **Claim**: Each t is in at least one closed interval $\overline{I^{t_j}}$, each $\overline{I^{t_j}}$ intersects at most two other $\overline{I^{t_k}}$'s, and the $\overline{I^{t_k}}$'s intersecting $\overline{I^{t_j}}$ do not intersect each other.

We will prove this by giving a recipe for choosing the sets. First, if $\overline{I^{t_1}}$ is contained in the union of two other intervals, then throw it out. Otherwise, consider the intervals whose left endpoint is in $\overline{I^{t_1}}$, find one whose right endpoint is largest and discard the others (which are already contained in these). Similarly, consider the intervals whose right endpoint is in $\overline{I^{t_1}}$ and throw out all but one whose left endpoint is smallest. Next, repeat this process on I^{t_2} (unless it has already been discarded), etc. After at most m steps, we get the desired cover, thus proving the claim.

For each $j = 1, \dots, m$, choose a continuous function $r_j : [0,1] \to [0,1]$ so that:

- $r_j(t) = 1$ on $\overline{I^{t_j}}$ and $r_j(t)$ is zero for $t \notin 2I^{t_j}$.
- $r_j(t)$ is zero on the intervals that *do not* intersect $\overline{I^{t_j}}$.

Property (1) follows directly and (2) follows from (5.108), (5.109), and (5.110). □

Proof of Theorem 5.23. Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ and $r_1, \dots, r_m : [0,1] \to [0, \pi)$ be given by Lemma 5.25. We will use an m step replacement process to define γ. Namely, first set $\gamma^0 = \tilde{\gamma}$ and then, for each $k = 1, \dots, m$, define γ^k by applying harmonic replacement to $\gamma^{k-1}(\cdot, t)$ on the k-th family of balls $r_k(t)\, \mathcal{B}_k$; i.e, set

$$(5.111) \qquad \gamma^k(\cdot, t) = H(\gamma^{k-1}(\cdot, t), r_k(t)\, \mathcal{B}_k) .$$

Finally, we set $\gamma = \gamma^m$.

A key point in the construction is that property (1) of the family of balls gives that only two $r_k(t)$'s are positive for each t. Therefore, the energy bound on the balls given by property (1) implies that each energy minimizing map replaces a map with energy at most $2\epsilon_1/3 < \epsilon_1$. Hence, Corollary 5.10 implies that these depend continuously on the boundary values, which are

themselves continuous in t, so that the resulting map $\tilde{\gamma}$ is also continuous in t. Finally, it is clear that $\tilde{\gamma}$ is homotopic to γ since continuously shrinking the disjoint closed balls on which we make harmonic replacement gives an explicit homotopy. Thus, $\gamma \in \Omega_{\tilde{\gamma}}$ as claimed.

For each t with $\mathrm{E}(\tilde{\gamma}(\cdot,t)) \geq W/2$, property (2) of the family of balls gives some $j(t)$ so that harmonic replacement for $\tilde{\gamma}(\cdot,t)$ on $\frac{r_j(t)}{2}\,\mathcal{B}_{j(t)}$ decreases the energy by at least $\frac{e_{\tilde{\gamma},\epsilon_1/8}(t)}{8}$. Thus, even in the worst case where $r_j(t)\,\mathcal{B}_{j(t)}$ is the second family of balls that we do replacement on at t, (5.37) in Lemma 5.11 gives

$$\mathrm{E}(\tilde{\gamma}(\cdot,t)) - \mathrm{E}(\gamma(\cdot,t)) \geq \kappa \left(\frac{e_{\tilde{\gamma},\epsilon_1/8}(t)}{8} \right)^2 . \tag{5.112}$$

To establish (B_Ψ), suppose that $\mathcal{B}$ is a finite collection of disjoint closed balls in $\mathbf{S}^2$ so that the energy of $\gamma(\cdot,t)$ on $\mathcal{B}$ is at most $\epsilon_1/12$. We can assume that $\gamma^k(\cdot,t)$ has energy at most $\epsilon_1/8$ on $\mathcal{B}$ for every k since otherwise Theorem 5.8 implies a positive lower bound for $\mathrm{E}(\tilde{\gamma}(\cdot,t)) - \mathrm{E}(\gamma(\cdot,t))$. Consequently, we can apply (5.38) in Lemma 5.11 twice (first with $\mu = 1/8$ and then with $\mu = 1/4$) to get

$$\begin{aligned} \mathrm{E}(\gamma(\cdot,t)) - \mathrm{E}\left[H(\gamma(\cdot,t),\frac{1}{8}\mathcal{B})\right] &\leq \mathrm{E}(\tilde{\gamma}(\cdot,t)) - \mathrm{E}\left[H(\tilde{\gamma}(\cdot,t),\frac{1}{2}\mathcal{B})\right] \\ &\quad + \frac{2}{\kappa}\left(\mathrm{E}(\tilde{\gamma}(\cdot,t)) - \mathrm{E}(\gamma(\cdot,t))\right)^{1/2} \\ &\leq e_{\tilde{\gamma},\frac{\epsilon_1}{8}}(t) + \frac{2}{\kappa}\left(\mathrm{E}(\tilde{\gamma}(\cdot,t)) - \mathrm{E}(\gamma(\cdot,t))\right)^{1/2} . \end{aligned} \tag{5.113}$$

Combining (5.112) and (5.113) with Theorem 5.8 gives (B_Ψ) and, thus, completes the proof. □

5.5. Existence of good sweepouts. The next result gives the existence of a sequence of good sweepouts.

Theorem 5.26 (Colding-Minicozzi, [**CM27**]). *Given a metric g on M and a map $\beta \in \Omega$ representing a nontrivial class in $\pi_3(M)$, there exists a sequence of sweepouts $\gamma^j \in \Omega_\beta$ with $\max_{s\in[0,1]} E(\gamma^j_s) \to W(g)$, and so that given $\epsilon > 0$, there exist $\bar{j}$ and $\delta > 0$ so that if $j > \bar{j}$ and*

$$\mathrm{Area}(\gamma^j(\cdot,s)) > W(g) - \delta\,, \tag{5.114}$$

then there are finitely many harmonic maps $u_i : \mathbf{S}^2 \to M$ with

$$d_V\left(\gamma^j(\cdot,s), \bigcup_i \{u_i\}\right) < \epsilon\,. \tag{5.115}$$

In (5.115), we have identified each map u_i with the varifold associated to the pair $(u_i, \mathbf{S}^2)$ and then taken the disjoint union of these $\mathbf{S}^2$'s to get

$\bigcup_i \{u_i\}$. The distance d_V in (5.115) is the varifold distance defined in (3.36) in Chapter 3.

Theorem 5.26 will be proven in subsection 5.6.

One immediate consequence of Theorem 5.26 is that if s_j is any sequence with Area$(\gamma^j(\cdot, s_j))$ converging to the width $W(g)$ as $j \to \infty$, then a subsequence of $\gamma^j(\cdot, s_j)$ converges to a collection of harmonic maps from $\mathbf{S}^2$ to M. In particular, the sum of the areas of these maps is exactly $W(g)$ and, since the maps are automatically conformal, the sum of the energies is also $W(g)$. The existence of at least one nontrivial harmonic map from $\mathbf{S}^2$ to M was first proven in [**SaUh**], but they allowed for loss of energy in the limit; cf. also [**St**]. Ruling out this possible energy loss in various settings is known as the "energy identity" and it can be rather delicate. This energy loss was ruled out by Siu and Yau, using also arguments of Meeks and Yau (see Chapter VIII in [**ScYa2**]). This was also proven later by Jost in theorem 4.2.1 of [**Jo**] which gives at least one min-max sequence converging to a collection of harmonic maps. The convergence in [**Jo**] is in a different topology that, as we will see, implies varifold convergence. See [**Pk**], [**DiTi**], [**Lt**], [**LfWc1**] and [**LfWc2**] for further results on the energy identity.

5.6. Constructing good sweepouts from the energy decreasing map on Ω. Given Theorem 5.23 and Proposition 5.12, we will prove Theorem 5.26. Let $\mathcal{G}^{W+1}$ be the set of collections of harmonic maps from $\mathbf{S}^2$ to M so that the sum of the energies is at most $W + 1$.

Proof of Theorem 5.26. Choose a sequence of maps $\tilde{\gamma}^j \in \Omega_\beta$ with

$$\max_{t \in [0,1]} \mathrm{E}\left(\tilde{\gamma}^j(\cdot, t)\right) < W + \frac{1}{j}, \tag{5.116}$$

and so that $\tilde{\gamma}^j(\cdot, t)$ is not harmonic unless it is a constant map.[9] We can assume that $W > 0$ since otherwise Area$(\tilde{\gamma}^j(\cdot, t)) \leq \mathrm{E}(\tilde{\gamma}^j(\cdot, t)) \to 0$ and the theorem follows trivially.

Applying Theorem 5.23 to the $\tilde{\gamma}^j$'s gives a sequence $\gamma^j \in \Omega_\beta$ where each $\gamma^j(\cdot, t)$ has energy at most that of $\tilde{\gamma}^j(\cdot, t)$. We will argue by contradiction to

[9] To do this, first use Lemma 5.21 (density of C^2-sweepouts) to choose $\tilde{\gamma}_1^j \in \Omega_\beta$ so $t \to \tilde{\gamma}_1^j(\cdot, t)$ is continuous from $[0,1]$ to C^2 and $\max_{t \in [0,1]} \mathrm{E}\left(\tilde{\gamma}_1^j(\cdot, t)\right) < W + \frac{1}{2j}$. Using stereographic projection, we can view $\tilde{\gamma}_1^j(\cdot, t)$ as a map from $\mathbb{R}^2$. Now fix a j. The continuity in C^2 gives a uniform bound $\sup_{t \in [0,1]} \sup_{B_1} |\nabla \tilde{\gamma}_1^j(\cdot, t)|^2 \leq C$ for some C. Choose $R > 0$ with $4\pi C R^2 \leq 1/(2j)$. Define a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ in polar coordinates by: $\Phi(r, \theta) = (2r, \theta)$ for $r < R/2$, $\Phi(r, \theta) = (R, \theta)$ for $R/2 \leq r \leq R$, and $\Phi(r, \theta) = (r, \theta)$ for $R < r$. Note that Φ is homotopic to the identity, is conformal away from the annulus $B_R \setminus B_{R/2}$, and on $B_R \setminus B_{R/2}$ has $|\partial_r \Phi| = 0$ and $|d\Phi| \leq 2$. It follows that $\tilde{\gamma}^j(\cdot, t) = \tilde{\gamma}_1^j(\cdot, t) \circ \Phi$ is in Ω_β, satisfies (5.116), and has $\partial_r \tilde{\gamma}^j(\cdot, t) = 0$ on $B_R \setminus B_{R/2}$. Since harmonic maps from $\mathbf{S}^2$ are conformal by Lemma 4.25, any harmonic $\tilde{\gamma}^j(\cdot, t)$ is constant on $B_R \setminus B_{R/2}$ and, thus, constant on $\mathbf{S}^2$ by unique continuation (theorem 1.1 in [**Sj**]).

show that the γ^j's have the desired property. Suppose, therefore, that there exist $j_k \to \infty$ and $s_k \in [0,1]$ with

$$d_V(\gamma^{j_k}(\cdot, s_k), \mathcal{G}^{W+1}) \geq \epsilon > 0\,,$$
$$\text{Area}(\gamma^{j_k}(\cdot, s_k)) > W - 1/k\,.$$

Thus, by (5.116) and the fact that $\text{E}(\cdot) \geq \text{Area}(\cdot)$, we get

$$\begin{aligned} \text{E}(\tilde{\gamma}^{j_k}(\cdot, s_k)) - \text{E}(\gamma^{j_k}(\cdot, s_k)) &\leq \text{E}(\tilde{\gamma}^{j_k}(\cdot, s_k)) - \text{Area}(\gamma^{j_k}(\cdot, s_k)) \\ &\leq 1/k + 1/j_k \to 0\,, \end{aligned} \tag{5.117}$$

and, similarly,

$$\text{E}\left(\gamma^{j_k}(\cdot, s_k)\right) - \text{Area}\left(\gamma^{j_k}(\cdot, s_k)\right) \to 0\,.$$

Using (5.117) in Theorem 5.23 gives

(B) If $\mathcal{B}$ is any collection of disjoint closed balls in $\mathbf{S}^2$ with $\int_{\bigcup_{\mathcal{B}} B} |\nabla \gamma^{j_k}(\cdot, s_k)|^2 < \epsilon_0$ and $v : \bigcup_{\mathcal{B}} \frac{1}{8}B \to M$ is an energy minimizing map that equals $\gamma^{j_k}(\cdot, s_k)$ on $\bigcup_{\mathcal{B}} \frac{1}{8}\partial B$, then

$$\int_{\bigcup_{\mathcal{B}} \frac{1}{8}B} \left|\nabla \gamma^{j_k}(\cdot, s_k) - \nabla v\right|^2 \leq \Psi(1/k + 1/j_k) \to 0\,. \tag{5.118}$$

Therefore, we can apply Proposition 5.12 to get that a subsequence of the $\gamma^{j_k}(\cdot, s_k)$'s varifold converges to a collection of harmonic maps. However, this contradicts the lower bound for the varifold distance to $\mathcal{G}^{W+1}$, thus completing the proof. □

5.7. Parameter spaces. Instead of using the unit interval, $[0,1]$, as the parameter space for the maps in the sweepout and assuming that the maps start and end in point maps, we could have used any compact finite dimensional topological space $\mathcal{P}$ and required that the maps are constant on $\partial\mathcal{P}$ (or that $\partial\mathcal{P} = \emptyset$). In this case, let $\Omega^{\mathcal{P}}$ be the set of continuous maps $\sigma : \mathbf{S}^2 \times \mathcal{P} \to M$ so that for each $t \in \mathcal{P}$ the map $\sigma(\cdot, t)$ is in $C^0 \cap W^{1,2}(\mathbf{S}^2, M)$, the map $t \to \sigma(\cdot, t)$ is continuous from $\mathcal{P}$ to $C^0 \cap W^{1,2}(\mathbf{S}^2, M)$, and finally σ maps $\partial\mathcal{P}$ to point maps. Given a map $\hat{\sigma} \in \Omega^{\mathcal{P}}$, the homotopy class $\Omega^{\mathcal{P}}_{\hat{\sigma}} \subset \Omega^{\mathcal{P}}$ is defined to be the set of maps $\sigma \in \Omega^{\mathcal{P}}$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega^{\mathcal{P}}$. Finally, the width $W = W(\hat{\sigma})$ is

$$\inf_{\sigma \in \Omega^{\mathcal{P}}_{\hat{\sigma}}} \max_{t \in \mathcal{P}} \text{E}\left(\sigma(\cdot, t)\right).$$

With only trivial changes, the same proof yields Theorem 5.26 for these general parameter spaces.[10]

[10]The main change is in Lemma 5.25 where the bound 2 for the multiplicity in (1) becomes $\dim(\mathcal{P}) + 1$. This follows from the definition of (covering) dimension; see pages 302–303 in [**Mu**].

Chapter 6

Embedded Solutions of the Plateau problem

After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

1. Unique Continuation

In studying the local properties of minimal surfaces, it will be useful to have a good description of their possible self-intersections and branch points. This can be achieved by using results about strong unique continuation for solutions of elliptic equations. The results of the next two sections will be used in the following sections to show first that minimal disks are immersed and then that (under additional hypotheses) they are embedded.

In particular, we will need the following quantitative version of strong unique continuation:

Theorem 6.1 (Quantitative Strong Unique Continuation). *Suppose that v is a nonconstant solution on D to*

$$0 = (a_{i,j}\, v_{x_j})_{x_i} + b_i\, v_{x_i} + c\, v\,, \tag{6.1}$$

where a_{ij} is symmetric, uniformly elliptic, and Lipschitz, and b_i, c are continuous. Then there exists some $\epsilon = \epsilon(a_{ij}, b_i, c) > 0$ and $\bar{d}$ such that for any

$2t < \epsilon$,

$$\frac{\int_{|x|=2t} v^2}{\int_{|x|=t} v^2} \leq 2^{2\bar{d}+1}\,. \tag{6.2}$$

In particular, there exists some integer $d \leq \bar{d}$ such that v vanishes precisely to order d at the origin.

In fact, it will follow from the proof of Theorem 6.1 that $\bar{d}$ may be estimated from above in terms of the growth of v from D_ϵ to $D_{1/2}$.

Remark 6.2. In particular, Theorem 6.1 implies that v cannot vanish to infinite order at any point. An operator which has this property is said to satisfy *strong unique continuation.* Note that if L has this strong unique continuation property, $Lv = 0$ on a connected domain Ω, and v vanishes on an open subset of Ω, then necessarily $v \equiv 0$ on Ω.

In the case where $a_{ij} = \delta_{ij}$ and $b_i = c = 0$, this theorem is closely related to Hadamard's three circles theorem from complex analysis.

1.1. Almgren's frequency function. We will first prove Theorem 6.1 in the case of the Laplacian on $\mathbb{R}^n$ using Almgren's frequency function; this proof can easily be modified to work in the general case.

Suppose that $\Delta\, u = 0$ on $\mathbb{R}^n$ and let $u_r = \frac{\partial u}{\partial r}$ denote its radial derivative. Given $r > 0$, define scale invariant quantities $I(r)$, $E(r)$, and the frequency function $U(r)$ (cf. Almgren, [**Am2**]) by

$$I(r) = r^{1-n} \int_{\partial B_r} u^2\,, \tag{6.3}$$

$$E(r) = r^{2-n} \int_{B_r} |\nabla u|^2 = r^{2-n} \int_{\partial B_r} u\, u_r\,, \tag{6.4}$$

$$U(r) = \frac{E(r)}{I(r)}\,. \tag{6.5}$$

Differentiating I gives

$$I'(r) = 2\, r^{1-n} \int_{\partial B_r} u\, u_r = \frac{2\, E(r)}{r}\,, \tag{6.6}$$

where the last equality used the divergence theorem since $\text{div}(u \nabla u) = |\nabla u|^2$ for the harmonic function u.

Lemma 6.3. *The derivative of E is*

$$E'(r) = 2\, r^{2-n} \int_{\partial B_r} |u_r|^2\,. \tag{6.7}$$

When u is harmonic on the two-dimensional disk, Lemma 6.3 can be proven by writing $u = \operatorname{Re} f$ (where f is holomorphic), using the Cauchy-Riemann equations, and integrating by parts.

Proof of Lemma 6.3. Using the coarea formula (i.e., (1.59)) to differentiate E gives

$$E'(r) = \frac{2-n}{r} E(r) + r^{2-n} \int_{\partial B_r} |\nabla u|^2 . \tag{6.8}$$

Set $r^2 = |x|^2$ on $B_r \subset \mathbb{R}^n$, so that the Hessian $(r^2)_{ij} = 2\,\delta_{ij}$ and observe that

$$\operatorname{div}\left[(r^2)_j u_j u_k - \frac{1}{2}\, u_j^2 (r^2)_k \right] = (2-n)\,|\nabla u|^2 + 2\, r\, u_r\, \Delta u . \tag{6.9}$$

Apply the divergence theorem to get

$$(2-n) \int_{B_R} |\nabla u|^2 = 2\,R \int_{\partial B_R} u_r^2 - R \int_{\partial B_R} |\nabla u|^2 . \tag{6.10}$$

Substituting (6.10) into the formula for $E'(r)$ gives the lemma. □

Lemma 6.4. *Differentiating* $\log U(r)$ *gives*

$$\frac{d}{dr} \log U(r) = \frac{2\, r^{-n}}{E(r)} \int_{\partial B_r} \left(r\, u_r - U(r)\, u \right)^2 \geq 0 . \tag{6.11}$$

Proof. Differentiating $\log U(r)$ gives and substituting the derivatives of I and D gives

$$\begin{aligned} \frac{d}{dr} \log U(r) &= \frac{E'(r)}{E(r)} - \frac{I'(r)}{I(r)} \\ &= \frac{2}{E(r)} \left(r^{2-n} \int_{\partial B_r} |u_r|^2 - \frac{E^2(r)}{r\, I(r)} \right) . \end{aligned} \tag{6.12}$$

Note that

$$\int_{\partial B_r} \left(r\, u_r - U(r)\, u \right)^2 = \int_{\partial B_r} r^2\, |u_r|^2 - \frac{r\, E^2(r)}{I(r)} . \tag{6.13}$$

Substituting this into (6.12) gives the lemma. □

Example 6.5. Suppose that $v(x)$ is a homogeneous harmonic polynomial of degree d. It is easy to see that $r\, v_r = d\, v$ and hence that $E(t) = d\, I(t)$. Consequently, in this case, $U(t) \equiv d$. More generally, suppose that $w = w_1 + w_2$ is a harmonic function where w_1 is homogeneous of degree d and the Taylor series of w_2 starts at degree $d+1$. Using the orthogonality of the spherical harmonics, we see that $U(t) \geq d$ in this case.

Although we will not use it, we note that if v is harmonic and the frequency is constant, then v must be homogeneous and hence a spherical harmonic. Namely, we have the following:

Lemma 6.6. *If v is harmonic on D and the frequency function $U(t)$ is constant (say $U(t) = d$), then v is a homogeneous harmonic polynomial of degree d.*

Proof. Since $U(t) = d$ is constant, Lemma 6.4 gives that

$$v_r(x) = d\,|x|^{-1}\,v(x)\,. \tag{6.14}$$

It follows from (6.14) that v is homogeneous of degree d and is hence a spherical harmonic. □

We are now prepared to prove Theorem 6.1 when $\Delta v = 0$. In fact, we will prove a stronger statement in this case which will cover Theorem 6.12 stated below.

Proof of Theorem 6.1 for $\Delta v = 0$. We will show this from the monotonicity of the frequency function. Differentiating $\log I(t)$ (see (6.6)) gives

$$(\log I)'(t) = 2\frac{U(t)}{t}\,. \tag{6.15}$$

Therefore, using the monotonicity of $U(t)$, we can write

$$\frac{I(1)}{I(\frac{1}{2})} = \mathrm{e}^{2\int_{\frac{1}{2}}^{1}\frac{U(t)}{t}\,dt} \geq \mathrm{e}^{2U(\frac{1}{2})\log 2} = 4^{U(\frac{1}{2})}\,. \tag{6.16}$$

Again using the monotonicity of U, we see by (6.16) that $U(t)$ is uniformly bounded for $t \leq \frac{1}{2}$. Consequently, integrating (6.15) and using the uniform bound on $U(t)$, we see that for $0 < 2s \leq t < \frac{1}{4}$,

$$\frac{I(2s)}{I(s)} = \mathrm{e}^{2\int_{s}^{2s}\frac{U(t')}{t'}\,dt'} \leq 2^{2U(2s)} \leq \frac{I(2t)}{I(t)}\,. \tag{6.17}$$

The estimate (6.17) is known as a doubling estimate. Iterating (6.17) bounds the order of vanishing at the origin (which gives quantitative strong unique continuation). Consequently, the Taylor polynomial for v at the origin does not vanish identically; let v^1 denote the first nonzero Taylor polynomial (which has degree $d \geq 1$). Since v is harmonic, v_1 and $v - v_1$ are harmonic. By construction, $v - v_1$ vanishes to order at least $d + 1$ and the theorem follows. □

1.2. Frequency functions for general operators. The key to proving this strong unique continuation theorem was obtaining the doubling estimate (6.17). This doubling estimate was an immediate consequence of a uniform bound on the frequency. When $\Delta v = 0$ on the disk, we deduced this bound on the frequency from the monotonicity of the frequency. For more general equations, as in Theorem 6.1, we can write down a frequency function $U(t)$, but it is not necessarily monotone. However, if the coefficients are sufficiently regular, then there will exist some constant Λ such that $e^{\Lambda t}\, U(t)$ is monotone. This gives enough control to establish a uniform bound on $U(t)$ for small t and hence the doubling estimate.

As an example, consider the case where $a_{ij} = f\,\delta_{ij}$ with $f(0) = 1$, $|\nabla f| \leq \lambda$, and $b_i = c = 0$. Suppose that v is a nonconstant solution on D to

$$\operatorname{div}(f\,\nabla v) = 0\,. \tag{6.18}$$

It will be clear to the reader how to modify these arguments when b_i and c do not vanish. Furthermore, equations of the form (6.18) (including the cases where b_i and c appear) will suffice for all of our applications.

Remark 6.7. As noted at the end of section 2 of [**GaLf**], equations of the form (6.1) can be transformed into the form (6.18) by making a suitable change of metric (see [**GaLf**] for details).

In general, the above transformation involves introducing a new metric and hence perturbation terms. In two dimensions, this can be avoided by using the existence of isothermal coordinates.

Example 6.8 (Laplacian for a Riemannian Metric)**.** If g_{ij} is a Riemannian metric in a coordinate chart in $\mathbb{R}^2$, the associated Laplacian Δ_g is given by

$$\Delta_g v = \frac{1}{\sqrt{g}}\,(\sqrt{g}\,g^{ij}\,v_{x_j})_{x_i}\,, \tag{6.19}$$

where g and g^{ij} are the determinant and inverse matrix of g_{ij}. The existence of isothermal coordinates allows us, after a change of coordinates, to suppose that $g_{ij} = f\,\delta_{ij}$ for a positive function f. In this case, (6.19) becomes

$$\Delta_g v = \frac{1}{f}\,v_{x_i\,x_i}\,. \tag{6.20}$$

For $0 < t \leq 1$, define $I(t)$ and $E(t)$ by

$$I(t) = t^{-1}\int_{|x|=t} f\,v^2\,. \tag{6.21}$$

Using Stokes' theorem and the formula $\operatorname{div}(f\,\nabla v^2) = 2f\,|\nabla v|^2$ (by (6.18)), we get

$$E(t) = \int_{|x|\leq t} f\,|\nabla v|^2 = \int_{|x|=t} f\,v_r\,v\,. \tag{6.22}$$

Once again, we define the frequency $U = I/E$ as in (6.5).

Henceforth, we assume that t, $|x| \leq \frac{1}{2\lambda}$ so that $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$. Differentiating $I(t)$ gives

$$I'(t) = 2t^{-1} \int_{|x|=t} f\, v\, v_r + t^{-1} \int_{|x|=t} f_r\, v^2 \,, \tag{6.23}$$

so Stokes' theorem and the bound $|\nabla f| \leq 2\lambda\, f$ give

$$\left| I'(t) - 2\frac{E(t)}{t} \right| \leq 2\,\lambda\, I(t) \,. \tag{6.24}$$

The definition of $U(t)$ and (6.24) give the differential inequality

$$\left| (\log I)'(t) - 2\frac{U(t)}{t} \right| \leq 2\,\lambda \,. \tag{6.25}$$

The coarea formula (i.e., (1.59)) gives

$$E'(t) = \int_{|x|=t} f\, |\nabla v|^2 \,. \tag{6.26}$$

Applying Stokes' theorem to (6.26) as before, we get

$$\begin{aligned} E'(t) &= \int_{|x|=t} f\, |\nabla v|^2 = t^{-1} \int_{|x|=t} \langle f\, |\nabla v|^2 x, |x|^{-1}\, x \rangle \\ &= 2t^{-1} E(t) + t^{-1} \int_{|x| \leq t} x_j\, (f v_{x_i} v_{x_i})_{x_j} \,, \end{aligned} \tag{6.27}$$

since

$$\operatorname{div}(f\, |\nabla v|^2 x) = (f\, v_{x_i}^2 x_j)_{x_j} = x_j\, (f v_{x_i} v_{x_i})_{x_j} + 2 f\, v_{x_i}^2 \,. \tag{6.28}$$

Since $(f v_{x_i})_{x_i} = 0$, $v_{x_i x_j} = v_{x_j x_i}$, and $(x_i)_{x_j} = \delta_{ij}$,

$$\begin{aligned} x_j (f v_{x_i} v_{x_i})_{x_j} &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 x_j f v_{x_i} v_{x_j x_i} \\ &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 (x_j f v_{x_i} v_{x_j})_{x_i} - 2 x_j v_{x_j} (f v_{x_i})_{x_i} - 2 f v_{x_i}^2 \\ &= x_j f_{x_j} v_{x_i} v_{x_i} + 2 (x_j f v_{x_i} v_{x_j})_{x_i} - 2 f v_{x_i}^2 \,. \end{aligned} \tag{6.29}$$

Substituting equation (6.29) into (6.27), integrating by parts, and using the bound $|\nabla f| \leq 2\,\lambda\, f$ yields

$$\left| E'(t) - 2 \int_{|x|=t} f\, v_r^2 \right| \leq 4\,\lambda\, E(t) \,. \tag{6.30}$$

Combining (6.25) and (6.30), differentiating $\log U(t)$ gives

$$\left| (\log U)'(t) - \frac{2}{E(t)} \int_{|x|=t} f\, v_r^2 + 2\frac{U(t)}{t} \right| \leq 6\,\lambda \,. \tag{6.31}$$

Using Cauchy-Schwarz gives

$$0 \leq \frac{2}{E(t)} \int_{|x|=t} f\, v_r^2 - 2\frac{U(t)}{t}\,. \tag{6.32}$$

Combining (6.31) and (6.32), we see that $e^{6\lambda t}\, U(t)$ is monotone nondecreasing. In particular, for all $0 < t \leq \frac{1}{2\lambda}$,

$$U(t) \leq e^{6\lambda t}\, U(t) \leq e^3\, U\left(\frac{1}{2\lambda}\right). \tag{6.33}$$

Combining the uniform bound on the frequency, (6.33), and the differential inequality (6.25), we obtain the desired bound on the order of vanishing of v. This completes the proof of Theorem 6.1 for equations of the form (6.18).

2. Local Description of Nodal and Critical Sets

Using the results of the previous section, we will obtain a useful local description of solutions to elliptic equations on the disk D. We will focus primarily on obtaining local asymptotic expansions and on the consequences of these expansions for the nodal and critical sets.

Definition 6.9 (Nodal and Critical Sets). Given a function u, the *nodal set* of u is the set $\{u = 0\}$ where u vanishes. The *critical set* of u is the set $\{u = 0\} \cap \{\nabla u = 0\}$ where u and ∇u both vanish.

Before we get to the main results of this section, we will need some background material on elliptic estimates. As we saw in Chapter 1, the mean value inequality bounds the maximum of a harmonic function by its average on a larger ball. Combining this with the reverse Poincaré inequality, one can then control all higher derivatives similarly. For more general equations, similar arguments give the following lemma (see theorem 8.8 of [**GiTr**] or [**HaLi**] for a proof):

Lemma 6.10. *Let v be a solution to* (6.1) *on D, and suppose further that a_{ij}, b_i, c are smooth. Then there exists constants $\epsilon > 0$ and $C < \infty$ which depend on a_{ij}, b_i, c such that for $|x| \leq \epsilon$,*

$$|v(x)|^2 + |x|^2\, |\nabla v(x)|^2 + |x|^4\, |\nabla^2 v(x)|^2 \leq C\, |x|^{-2} \int_{B_{2\,|x|}} v^2\,. \tag{6.34}$$

Combining Theorem 6.1 and Lemma 6.10 gives the following corollary:

Corollary 6.11. *Let v be a solution to* (6.1) *on D, and let $\bar{d}$ be the bound on the order of vanishing given by Theorem* 6.1. *Suppose further that a_{ij}, b_i, and c are smooth. Then there exists constants $\epsilon > 0$ and $C < \infty$ which depend on a_{ij}, b_i, c and $\bar{d}$ such that for $|x| \leq \epsilon$,*

$$|v(x)|^2 + |x|^2\, |\nabla v(x)|^2 + |x|^4\, |\nabla^2 v(x)|^2 \leq C\, I(|x|)\,. \tag{6.35}$$

Proof. By Theorem 6.1, the frequency $U(t)$ is uniformly bounded by $\bar{d}$ on some ball $\{|x| \leq 2\,\epsilon\}$. Consequently, we have for any $s \leq \epsilon$ that

$$I(2s) \leq 2^{2\bar{d}}\, I(s)\,. \tag{6.36}$$

Integrating the bound (6.36) (and using the trivial monotonicity of $I(t)$) gives

$$\int_{|x|\leq 2s} v^2 = \int_{t=0}^{2s} \int_{|x|=t} v^2\, dt \leq \int_{t=0}^{2s} t\, I(t)\, dt \leq C_1\, s^2\, I(s)\,. \tag{6.37}$$

The estimate (6.35) now follows by combining (6.34) and (6.37). □

Theorem 6.12 (Local Asymptotics, [**Ber**]). *Let v be a solution to* (6.1), *let $\bar{d}$ and d be the bound on the order of vanishing and the order of vanishing at* 0, *respectively, given by Theorem* 6.1. *Suppose further that a_{ij}, b_i, and c are smooth. There exists a linear transformation T such that after rotating by T we can write*

$$v(x) = p_d(x) + q(x)\,, \tag{6.38}$$

where p_d is a homogeneous harmonic polynomial of degree d and

$$|q(x)| + |x|\, |\nabla q(x)| + \cdots + |x|^d\, |\nabla^d q(x)| \leq C|x|^{d+1} \tag{6.39}$$

for some $C < \infty$ which depends on a_{ij}, b_i, c and $\bar{d}$.

Proof. After a linear transformation, we may assume that $a_{ij}(0) = \delta_{ij}$ and $a_{ij}(x) - \delta_{ij} = c_{ij}(x)$ where $|c_{ij}(x)| \leq \lambda\, |x|$. We can then write (6.1) as

$$\begin{aligned} \Delta v &= -(c_{i,j}\, v_{x_j})_{x_i} - b_i\, v_{x_i} - c\, v \\ &= -c_{i,j}\, v_{x_i\, x_j} - b_i'\, v_{x_i} - c\, v\,, \end{aligned} \tag{6.40}$$

where $b_i' = b_i + (c_{i,j})_{x_j}$.

By Theorem 6.1, $v(x)$ vanishes precisely to order d at the origin. Therefore the limit

$$\lim_{t\to 0} t^{-2d}\, I(t) \tag{6.41}$$

exists and is not zero. In fact, if $p_d(x)$ denotes the degree d Taylor polynomial for v at the origin, then

$$\lim_{t\to 0} t^{-2d}\, I(t) = \lim_{t\to 0} t^{-2d-1} \int_{|x|=t} p_d^2\,. \tag{6.42}$$

Define the error term q by $q(x) = v(x) - p_d(x)$ so that q vanishes to order at least $d+1$ at the origin.

Since v satisfies (6.40), we have

$$\lim_{|x|\to 0} |x|^{d-2} \Delta v = - \lim_{|x|\to 0} |x|^{d-2} \left(c_{i,j}\, v_{x_i\, x_j} - b_i'\, v_{x_i} - c\, v\right)\,. \tag{6.43}$$

The bound on the frequency implies that the estimate (6.35) applies to v. Hence the limits of the second two terms in (6.43) are both zero and $v_{x_i x_j}$ is of order $|x|^{d-2}$. However, $|c_{ij}| \leq \lambda\,|x|$ so the last term on the right-hand side also goes to zero. Consequently,

$$|\Delta v| \leq C\,|x|^{d-1}\,. \tag{6.44}$$

On the other hand, since $v - p_d$ is of order at least $d+1$ the Hessian $\nabla^2(v - p_d)$ is of order at least $d-1$. Therefore (6.44) implies that Δp_d is of order at least $d-1$.

However, p_d is homogeneous of degree d and hence Δp_d is a homogeneous degree $d-2$ polynomial. A homogeneous degree $d-2$ polynomial which is of order at least $d-1$ must vanish identically, and hence p_d is a spherical harmonic.

Substituting the definition of q into (6.40), we see that q itself satisfies an elliptic equation

$$\Delta q = \Delta(v - p_d) = \Delta v = -c_{i,j}\, v_{x_i x_j} - b_i'\, v_{x_i} - c\, v\,. \tag{6.45}$$

As we have already verified, the right-hand side of (6.45) vanishes to order at least $d+1$. Lemma 6.10 then yields the higher-order estimates in (6.39). □

We will use Theorem 6.12 repeatedly to describe the local behavior of minimal surfaces and their almost conformal parameterizations.

In two dimensions, a spherical harmonic $p_d(x)$ of degree d can be written as a constant times the real part of $\mathrm{e}^{i\,\theta}\,(x_1 + i\,x_2)^d$ for some $\theta \in [0, 2\pi)$. The real part of a complex vector-valued function f will be denoted by $\operatorname{Re} f$.

Lemma 6.13. *Suppose that*

$$v(x) = p_d(x) + q(x)\,, \tag{6.46}$$

where p_d is a nontrivial homogeneous harmonic polynomial of degree d, and there is a constant C such that

$$|q(x)| + |x|\,|\nabla q(x)| \leq C|x|^{d+1}\,; \tag{6.47}$$

then there exists a neighborhood U of 0 and a C^1 diffeomorphism $F : U \to D$ with $F(0) = 0$ such that $F(U \cap \{v = 0\})$ is equal to the $2d$ smooth arcs given by

$$\alpha_j(t) = t\,\mathrm{e}^{\frac{j\,\pi\,i}{d}} \tag{6.48}$$

for $t \in [0, \epsilon]$ and $j = 1, \ldots, 2d$.

Proof. Since we are in two dimensions, $p_d(x)$ can be written as a constant times the real part of $e^{i\theta} z^d$ where $z = x_1 + i\,x_2$. After a rotation, we may suppose that $\theta = 0$ so that

$$v(z) = \mathrm{Re}(z^d + q(z))\,. \tag{6.49}$$

The estimate (6.47) implies that the map

$$F(z) = z(1 + z^{-d}\,q(z))^{\frac{1}{d}} \tag{6.50}$$

is well defined for $|z| \leq \frac{1}{2C}$. The complex derivative of F is given by

$$\begin{aligned} F_z(z) &= (1 + z^{-d}\,q(z))^{\frac{1}{d}} \\ &\quad + \frac{z}{d}(1 + z^{-d}\,q(z))^{\frac{1-d}{d}}(z^{-d}\,q_z(z) - d\,z^{-d-1}\,q(z))\,. \end{aligned} \tag{6.51}$$

Applying (6.47) again, we see that F is C^1 and $F_z(0) = 1$. The implicit function theorem gives the desired neighborhoods and a C^1 inverse F^{-1}.

Composing with the diffeomorphism F, we see that $v(x) = \mathrm{Re}(F(z))^d$. The description (6.48) follows immediately. □

2.1. Rado's theorem. The first application of Theorem 6.12 of Bers will be to obtain a result of Rado.

Theorem 6.14 (Rado, [**Ra2**]). *Suppose that $\Omega \subset \mathbb{R}^2$ is a convex subset and $\sigma \subset \mathbb{R}^3$ is a simple closed curve which is graphical over $\partial\Omega$. Then any minimal disk $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique by the maximum principle.*

Proof. Suppose that Σ is such a minimal disk that is not graphical. Then the projection to the (x_1, x_2)-plane cannot be an immersion. Consequently, there exists some $x \in \Sigma$ such that at x $\nabla_\Sigma x_1$ and $\nabla_\Sigma x_2$ are linearly dependent. In other words, there exists $(a, b) \neq (0, 0)$ such that

$$\nabla_\Sigma(a\,x_1 + b\,x_2)(x) = 0\,. \tag{6.52}$$

By Proposition 1.7, $a\,x_1 + b\,x_2$ is harmonic on Σ (since it is a linear combination of coordinate functions). We may therefore apply Theorem 6.12, and it follows from (6.52) and Lemma 6.13 that the nodal line

$$a\,x_1 + b\,x_2 = (a\,x_1 + b\,x_2)(x) \tag{6.53}$$

has a singularity at x where at least four different curves meet. If two of these nodal curves were to meet again, then there would be a closed nodal curve which must bound a disk (since Σ is a disk). By the maximum principle, $a\,x_1 + b\,x_2$ would have to be constant on this disk and hence constant on Σ by unique continuation. This would imply that $\sigma = \partial\Sigma$ is contained in the plane given by (6.53). Since this is impossible, we conclude that all of these curves go to the boundary without intersecting again.

In other words, the plane in $\mathbb{R}^3$ given by (6.53) intersects σ in at least four points. However, since $\Omega \subset \mathbb{R}^2$ is convex, $\partial\Omega$ intersects the line given by (6.53) in exactly two points. Finally, since σ is graphical over $\partial\Omega$, σ intersects the plane in $\mathbb{R}^3$ given by (6.53) in exactly two points, which gives the desired contradiction. □

2.2. Normal forms for harmonic maps. We will next apply Theorem 6.12 to obtain a normal form for a harmonic map $u : D \to \mathbb{R}^3$.

Lemma 6.15. *Let $u : D \to \mathbb{R}^3$ be a nonconstant harmonic map with $u(0) = 0$ and $\nabla u(0) = 0$. The following local asymptotic expansion holds:*

$$u(x) = \mathrm{Re} \sum_{j=d}^{2d-1} a_j (x_1 + i x_2)^j + q(x) , \tag{6.54}$$

where $d \geq 2$, each $a_i \in \mathbf{C}^3$, $a_d \neq 0$, and

$$|q(x)| + |x|\,|\nabla q(x)| \leq C|x|^{2d} . \tag{6.55}$$

Proof. Since the components of u are actually harmonic functions on D, Theorem 6.1 implies that u vanishes to some finite order $d \geq 2$ there. Let u^1 denote the first d terms in this expansion (so that u^1 has terms of degree d to $2d-1$ and the degree d term is nontrivial). We can then write

$$u(x) = u^1(x) + q(x)$$

where $q(x)$ satisfies (6.55), and the claim follows. □

2.3. Branch points. Suppose now that $u : D \to \mathbb{R}^3$ is an almost conformal harmonic map (so that the image is minimal). We saw in Chapter 4 that u is an immersion away from the (isolated) branch points where u_{x_1} and u_{x_2} both vanish. We are interested in giving a description of u near such a branch point. Our presentation will roughly follow the arguments of Gulliver in [**Gu**].

We let $z = x_1 + i x_2$ be the complex coordinate on D. Since u is almost conformal,

$$\langle u_z, u_z \rangle = 0 . \tag{6.56}$$

Note that this gives two equations (namely, that the real and imaginary parts both vanish). Applying this to the local representation (6.54), we get that $\langle a_d, a_d \rangle = 0$ and $\langle a_d, a_{d+1} \rangle = 0$. This first condition implies that the real and imaginary parts of a_d are orthogonal and of equal length. After a rotation, we can assume that $a_d = (a, -i\,a, 0)$ where $a > 0$. After dilating, we can take $a = 1$ so that $a_d = (1, -i, 0)$. Since $\langle a_d, a_{d+1} \rangle = 0$, we can write $a_{d+1} = (c, -i\,c, c')$. Substituting this back into (6.55), we get

$$u_1(z) + i u_2(z) = z^d + c z^{d+1} + G(z) , \tag{6.57}$$

where $d \geq 2$ and

$$|z|\,|u_3(z)| + |G(z)| + |z|\,|\nabla G(z)| \leq C\,|z|^{d+2}\,. \tag{6.58}$$

Equations (6.57) and (6.58) give a sort of normal form for minimal surfaces near a branch point. It will be convenient to obtain an even more canonical representation by choosing a nice reparameterization of D.

Lemma 6.16. *Suppose that* (6.57) *and* (6.58) *hold. Then there exist neighborhoods U and V of $0 \in D$ and a C^1 diffeomorphism $F : V \to U$ such that for $z \in V$,*

$$u_1 + iu_2 = (F(z))^d \tag{6.59}$$

and $u_3 = \phi(F(z))$, where

$$|\phi(z)| + |z|\,|\nabla\phi(z)| + |z|^2\,|\nabla^2\phi(z)| \leq C|z|^{d+1}\,. \tag{6.60}$$

If u is real analytic, then F and ϕ are real analytic away from 0.

Note that C in (6.60) is not the same as in (6.58). The fact that we can take F and ϕ to be real analytic will not be used until we study false branch points.

Proof. The estimate (6.58) on $G(z)$ implies that for small z,

$$|z|^{-d}\,|G(z)| < 1\,. \tag{6.61}$$

We can therefore define

$$F(z) = z(1 + z^{-d}\,G(z))^{\frac{1}{d}}\,. \tag{6.62}$$

Clearly, with this definition (6.59) is satisfied.

The estimate (6.58) implies that F is C^1 and the derivative at the origin is the identity. Consequently, the implicit function theorem gives the desired neighborhoods U and V and a C^1 inverse F^{-1}. The estimate (6.60) follows immediately from (6.58).

If u is real analytic, then so is G. It follows immediately from (6.62) that F is real analytic away from 0. Since ϕ is the composition of a real analytic function with F it is real analytic where F is (namely, away from 0). □

The image minimal surface is locally a multi-valued graph over the plane $x_3 = 0$. We will next analyze the height function ϕ. Taking into account the parameterization (6.59), we can write the minimal surface equation for ϕ from Lemma 6.16 as

$$\operatorname{div}\left(\frac{\nabla\phi}{W}\right) = 0\,, \tag{6.63}$$

where

$$W = \left(1 + \frac{|\nabla \phi|^2}{d^2|z|^{2d-2}}\right)^{\frac{1}{2}}. \tag{6.64}$$

When $d = 1$, (6.63) is the classical minimal surface equation (described in Chapter 1). For $d \geq 2$, (6.60) implies that W is bounded so that (6.63) is again a uniformly elliptic equation.

Lemma 6.17. *Suppose that $d \geq 1$ is fixed, ϕ_1 and ϕ_2 are solutions on D to (6.63) with W as in (6.64), $\phi_i(0) = 0$, and $|\phi_i(z)| \leq C_0\,|z|^{d+1}$. If $\Phi = \phi_1 - \phi_2$ is not identically zero, then there exists an integer $n \geq d+1$ (in fact, $n \geq d+2$) and an asymptotic expansion*

$$\Phi(z) = \operatorname{Re} c\, z^n + \rho(z)\,, \tag{6.65}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla \rho(z)| \leq C|z|^{n+\epsilon} \tag{6.66}$$

for some $\epsilon > 0$ and $C < \infty$.

Proof. As in the proof of the strong maximum principle (Lemma 1.27), we conclude that Φ satisfies an equation of the form

$$0 = (a_{i,j}\,\Phi_{x_j})_{x_i} + b_i\,\Phi_{x_i} + c\,\Phi\,, \tag{6.67}$$

where a_{ij}, b_i, c are smooth, a_{ij} is symmetric, uniformly elliptic, and $a_{ij}(0) = \delta_{ij}$.

We are now in a position to apply Theorem 6.12 to Φ. We conclude that if Φ does not vanish identically, then there is an asymptotic expansion of the form (6.65) since $\Phi(0) = 0$. □

The cases $d \geq 2$ of Lemma 6.17 will be applied to study branch points of minimal surfaces. In later sections, we will apply Lemma 6.17 to the minimal surface equation (that is, the case $d = 1$) when we analyze the intersections of immersed minimal surfaces.

Suppose now that ζ is a d-th root of unity (that is, $\zeta^d = 1$). Then $\phi_\zeta(z) \equiv \phi(\zeta\, z)$ also satisfies (6.63). Moreover, the self-intersections of the minimal surface are the points where ϕ_ζ and ϕ agree (as we allow all possible values of ζ). Fixing ζ for now, define the function Φ by

$$\Phi(z) = \phi_\zeta(z) - \phi(z)\,. \tag{6.68}$$

By Lemma 6.17, if Φ is not identically zero, we get an asymptotic expansion of the form

$$\Phi(z) = \operatorname{Re} c\, z^n + \rho(z)\,, \tag{6.69}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla\rho(z)| \leq C|z|^{n+\epsilon} \tag{6.70}$$

for some $\epsilon > 0$.

Arguing exactly as in Lemma 6.16, we can make a C^1 change of coordinates F which is conformal at the origin such that on some ball B_{r_0} we can represent Φ as

$$\Phi(z) = \operatorname{Re} F(z)^n\,. \tag{6.71}$$

Finally, the representation (6.71) allows us to give a good description of u near a branch point.

Theorem 6.18. *Let $u : D \to \mathbb{R}^3$ be an almost conformal harmonic map (so that the image is minimal). There exists some neighborhood V containing the origin such that either $u(V)$ is an immersed surface or there exist simple C^1 arcs $\gamma_i : [0,1] \to D$ with $\gamma_i(0) = 0$, $|\gamma_i'(0)| = 1$, $\gamma_1'(0) \neq \gamma_2'(0)$, $u(\gamma_1(t)) = u(\gamma_2(t))$ for all t, and such that the tangent spaces to the image of u meet transversely along the images of the γ_i.*

Proof. We may suppose that u has a branch point at 0 (otherwise the first option holds trivially). By Lemma 6.16, there is a neighborhood V such that, after a coordinate change, we have the representation $u_1 + i\,u_2 = z^n$ and $u_3(z) = \phi(z)$ where (6.60) holds. Let $\zeta = e^{2\pi i/n}$ and $\Phi = \phi_\zeta - \phi$.

If Φ is identically zero, then $\phi(z) = \phi(\zeta\, z)$ for all z and hence $\phi(z) = \bar{\phi}(z^n)$ for a smooth function $\bar{\phi}$. We can then reparametrize the image $u(V)$ by taking $\bar{u}(z) = u(z^{1/n})$. The image of $\bar{u}$ is an immersed surface and it clearly coincides with the image of u.

Otherwise, if Φ does not vanish identically, then we have the representation (6.71) for Φ in the coordinates given by F. In these coordinates,

$$\Phi(t\,e^{\frac{\pi i}{2n}}) = \operatorname{Re}(i\,t^n) = 0 \tag{6.72}$$

for $0 \leq t \leq r_0$. Since, by definition, Φ vanishes when ϕ and ϕ_ζ agree, (6.72) implies that

$$\phi(t\,e^{\frac{\pi i}{2n}}) = \phi_\zeta(t\,e^{\frac{\pi i}{2n}}) = \phi(t\,e^{\frac{5\pi i}{2n}}) \tag{6.73}$$

for $0 \leq t \leq r_0$. Therefore, in the coordinates given by F, the curves γ_i are given by the rays from the origin in B_{r_0} with angles $\frac{\pi}{2n}$ and $\frac{5\pi}{2n}$.

Finally, since $\{|\nabla \operatorname{Re} z^n| = 0\} = \{0\}$, the representation (6.71) implies that $\nabla\Phi$ does not vanish along the γ_i (except at the origin). Consequently, the tangent spaces to the image of u meet transversely along the images of the γ_i. □

Definition 6.19. If 0 is a branch point of u but there exists some neighborhood V containing the origin such that $u(V)$ is an immersed surface, then we say that 0 is a *false branch point*. If there exists some neighborhood V containing the origin and simple C^1 arcs $\gamma_i : [0,1] \to D$ with $\gamma_i(0) = 0$, $|\gamma_i'(0)| = 1$, $\gamma_1'(0) \neq \gamma_2'(0)$, $u(\gamma_1(t)) = u(\gamma_2(t))$ for all t, and such that the tangent spaces to the image of u meet transversely along the images of the γ_i, then 0 is a *true branch point*.

False branch points come from the parameterization whereas true branch points are visible in the geometry of the image surface. For this reason, it is significantly easier to rule out true branch points for area-minimizing disks. We will return to this in the next section.

3. Absence of True Branch Points

In this section, we will prove that a least area map $u : D \to \mathbb{R}^3$ does not have any true branch points; in other words, the image of u must be an immersed surface. In particular, if u is a solution to the Plateau problem (see Definition 4.15), then $u(D)$ is an immersed surface. True branch points were ruled out by R. Osserman in [**Os1**]. This leaves open the possibility that u has false branch points (where the problem is with the parameterization and not the surface). This will be addressed in the next section.

Theorem 6.20 (Osserman, [**Os1**]). *If $u : D \to \mathbb{R}^3$ is an almost conformal harmonic map with a true branch point at the origin, then there exists a map $v : D \to \mathbb{R}^3$ such that u and v have the same image, u and v are homeomorphic on ∂D, and v is not stationary for the energy functional with respect to some compactly supported smooth variation.*

As an immediate consequence, we see that if u has a true branch point it cannot be area-minimizing.

Corollary 6.21 (Osserman, [**Os1**]). *If $u : D \to \mathbb{R}^3$ is a solution to the Plateau problem, then it cannot have any true branch points.*

Proof. If u had a true branch point, then Theorem 6.20 would give a new map with the same energy which is not minimizing. This gives a contradiction. □

In the proof of Theorem 6.20, we will use the existence of a true branch point to construct a map (via cut-and-paste arguments) with the same energy which now folds along a curve. Since stationary maps from the disk have isolated singularities, the new map cannot be stationary.

Proof of Theorem 6.20. As promised we will construct an almost conformal map v with the same image as u so that

$$E_v = \text{Area}_v = \text{Area}_u = E_u \tag{6.74}$$

and such that v is not stationary for the energy. Before doing this, we will construct a map $\bar{v}$ with the same image as u that is continuous and is an immersion away from a set of measure zero. Morrey's version of the uniformization theorem guarantees that we can reparametrize $\bar{v}$ to get the almost conformal map v.

Suppose that 0 is a true branch point of $u : D \to \mathbb{R}^3$ and let γ_i, $i = 1, 2$, and V be given by Theorem 6.18. We may suppose that ∂V is smooth and that each curve γ_i intersects ∂V transversally. In fact, after reparameterizing the γ_i, we can arrange that

$$\gamma_i \cap \partial V = \gamma_i(\epsilon) \tag{6.75}$$

for some $\epsilon > 0$.

Choose a homeomorphism $\bar{F} : \overline{D_\epsilon} \to \overline{V}$ which is C^2 away from the origin with $\bar{F}(it) = \gamma_1(t)$ and $\bar{F}(-it) = \gamma_2(t)$ for $0 \leq t \leq \epsilon$.

We will next construct a map $G : \overline{D_\epsilon} \to \overline{D_\epsilon}$ with the following properties. First, G is the identity on ∂D_ϵ and continuous in a neighborhood of ∂D_ϵ. Second, for $-\frac{\epsilon}{2} \leq t \leq \frac{\epsilon}{2}$,

$$\lim_{s \to 0_+} G(t + is) = 2\, i\, |t| \tag{6.76}$$

and

$$\lim_{s \to 0_-} G(t + is) = -2\, i\, |t|\, . \tag{6.77}$$

Third, even though G will not be continuous across the real axis, $u(\bar{F}(G))$ is continuous everywhere. This is possible since (6.76) and (6.77) imply that $u(\bar{F}(G(t))) = u(\gamma_i(|t|))$. Fourth, $u(\bar{F}(G))$ is piecewise C^2 and an immersion almost everywhere.

There is clearly a great deal of freedom in constructing the map G. We describe one possible construction. Choose a discontinuous map $G : \overline{D_\epsilon} \to \overline{D_\epsilon}$ such that the negative and positive parts of the imaginary axis are mapped to $-i\epsilon$ and $i\epsilon$, respectively, $-\frac{\epsilon}{2}$ and $\frac{\epsilon}{2}$ are taken to the origin, the segments of discontinuity $[-\frac{\epsilon}{2}, 0]$ and $[0, \frac{\epsilon}{2}]$ are mapped according to (6.76) and (6.77), G is the identity on ∂D_ϵ and continuous on a neighborhood of ∂D_ϵ, and G is a diffeomorphism on each connected component of

$$D_\epsilon \setminus \left([-\frac{\epsilon}{2}, 0] \cup [0, \frac{\epsilon}{2}] \cup [-i\epsilon, i\epsilon] \right)\, . \tag{6.78}$$

The map G was constructed so that $u(\bar{F}(G))$ is continuous and piecewise C^2.

Define a new map $\bar{v}$ by

$$\begin{aligned}(6.79)\qquad \bar{v}(z) &= u(\bar{F}(G(\bar{F}^{-1}(z)))) \text{ for } z \in V \text{ and}\\ \bar{v}(z) &= u(z) \text{ for } z \notin V\,.\end{aligned}$$

The map $\bar{v}$ is continuous, piecewise C^2 and is an immersion away from a set of measure zero.

Morrey's version of the uniformization theorem (see Remark 4.3) implies that there is homeomorphism $T : D \to D_\epsilon$ in W^2 (so that its Hessian is in L^2) such that $v = \bar{v} \circ T$ is almost conformal. By construction, the images of u and v are identical so that they have equal area.

However, the transversality of the image minimal surface along the γ_i implies that the map v now has branch points along two entire segments. Therefore v cannot be stationary for energy since, by Corollary 4.18, the branch points for a stationary map are isolated. □

4. Absence of False Branch Points

The main result of this section, Theorem 6.23, shows that a solution to the Plateau problem $u : D \to \Omega \subset \mathbb{R}^3$, where Ω is mean convex and $u(\partial D)$ is embedded, does not have any false branch points. Combined with Osserman's result from the previous section, it follows that the energy-minimizing map u is an immersion.

The most general version of this result is due to R. Gulliver, theorem 8.2 of [**Gu**] (see also H. W. Alt [**Alt**]). Recall that u is said to be a solution to the Plateau problem with boundary Γ if u minimizes energy among all maps whose restriction to ∂D is a monotone map to Γ (see Definition 4.15).

Theorem 6.22 (Absence of False Branch Points, [**Gu**]; cf. [**Alt**]). *Let M^3 be a three-dimensional C^3 Riemannian manifold and $\Gamma \subset M$ a piecewise C^1 Jordan curve. If $u : D \to \mathbb{R}^3$ is a solution of the Plateau problem with boundary Γ, then u is an immersion on D. That is, u has no interior branch points.*

Since our primary interest will be in the special case where the boundary of $u(D)$ lies in a mean convex domain, we will not further discuss Theorem 6.22. The question of boundary branch points is not completely solved except in the case of real analytic boundary by Gulliver and F. Leslie, [**GuLe**]. On the other hand, the definitive boundary regularity result was proven for minimizing currents by Hardt and Simon in [**HaSi**] (see [**Wh2**] for the latest developments in the study of boundary regularity for minimal surfaces).

The main theorem of this section is the following:

Theorem 6.23 (Absence of False Branch Points). *Let Ω be a bounded mean convex region with smooth boundary and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is a solution of the Plateau problem as in Definition 4.15. If $u(\partial D) \subset \partial\Omega$ is embedded and $u(D) \cap \Omega \neq \emptyset$, then $u(D) \subset \Omega$ and u has no false branch points.*

Let $u : D \to \Omega \subset \mathbb{R}^3$ be a solution to the Plateau problem so that u is almost conformal, harmonic, and $u : \partial D \to \partial\Omega$ is monotone. In Corollary 6.21 (in the previous section), we saw that the image $u(D)$ is an immersed surface.

By Lemma 6.16, after a C^1 change of coordinates which is real analytic away from 0, there exists a neighborhood V of $0 \in D$ and an integer $d < \infty$ such that for $z \in V$,

$$u_1 + iu_2 = z^d \tag{6.80}$$

and $u_3 = \phi(z)$, where

$$|\phi(z)| + |z|\,|\nabla\phi(z)| + |z|^2\,|\nabla^2\phi(z)| \leq C|z|^{d+1}\,. \tag{6.81}$$

If $d \geq 2$, then 0 is a branch point for u. In this case, 0 is a false branch point if

$$\phi(z) \equiv \phi(\zeta_d\, z) \tag{6.82}$$

for all $z \in V$ and for any d-th root of unity ζ_d. By the results of the previous section, all interior branch points are false.

Of course, if 0 is not a branch point, then we have (6.80) and (6.81) with $d = 1$.

Before proving the theorem, we will need some preliminary lemmas.

Lemma 6.24. *Given any $z \in \bar{D}$ and compact set $K \subset D$, the set $K \cap u^{-1}(u(z))$ is finite.*

Proof. For any point $z_j \in K \cap u^{-1}(u(z)) \subset D$, the representation (6.80) implies there is a neighborhood V_{z_j} of z_j such that

$$V_{z_j} \cap u^{-1}(u(z)) = \{z_j\}\,. \tag{6.83}$$

Since u is continuous on $\bar{D}$, the set $u^{-1}(u(z))$ is compact and so is $K \cap u^{-1}(u(z))$. Therefore, there is a finite set of distinct points $z_1, \ldots, z_k$ such that

$$K \cap u^{-1}(u(z)) \subset \bigcup_{j=1}^{k} V_{z_j}\,. \tag{6.84}$$

Combining (6.83) and (6.84), we have $K \cap u^{-1}(u(z)) = \{z_j\}_{j=1,\ldots,k}$ and the lemma follows. □

We will now specialize to the case where the boundary of the minimal surface is contained in the boundary of a mean convex region. This situation will be the focus of the next section.

If the region is actually convex, then things are simpler, as the following example illustrates:

Example 6.25. Suppose that $u : \bar{D} \to \bar{B}_1 \subset \mathbb{R}^3$ is a continuous (on $\bar{D}$) harmonic map (not necessarily almost conformal) with $u(\partial D) \subset \partial B_1$ and $u(D) \cap B_1 \neq \emptyset$. Define $v : \bar{D} \to \mathbb{R}$ by $v(z) = 1 - |u(z)|^2$. Since $v \geq 0$, $v(\partial D) = 0$, and $\Delta v \leq 0$, the strong maximum principle implies that v is positive in D. In other words, $u(D) \subset B_1$. Furthermore, the Hopf boundary point lemma implies that the normal derivative to v does not vanish on ∂D. If u is, in addition, almost conformal and C^1 on $\bar{D}$, this nonvanishing implies that there are no branch points in a neighborhood of ∂D.

In the next lemma (Lemma 6.26), we will generalize this example to the case where $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is an almost conformal harmonic map with $u(\partial D) \subset \partial\Omega$ and $\partial\Omega$ is mean convex. To do this we will need to compute the Hessian of the distance function to the boundary of a mean convex region.

Let $\Omega \subset \mathbb{R}^3$ be a mean convex region in $\mathbb{R}^3$ and let ρ denote the distance to the boundary, that is,

$$\rho(x) = \operatorname{dist}(\partial\Omega, x)\,. \tag{6.85}$$

For $y \in \partial\Omega$, let $\kappa_1(y) \leq \kappa_2(y)$ denote the principal curvatures of $\partial\Omega$. Since the boundary $\partial\Omega$ is mean convex, we have

$$0 \leq \min_{y \in \partial\Omega} [\kappa_1(y) + \kappa_2(y)]\,. \tag{6.86}$$

Since $\partial\Omega$ is smooth and compact,

$$\max_{y \in \partial\Omega} [|\kappa_1(y)| + |\kappa_2(y)|] \leq \bar{\kappa} < \infty\,. \tag{6.87}$$

Combining (6.86) and (6.87), for any $y \in \partial\Omega$ and $t < 1/\bar{\kappa}$, we get

$$\begin{aligned} 0 &\leq \frac{\kappa_1(y) + \kappa_2(y) - 2\,t\,\kappa_1(y)\,\kappa_2(y)}{(1 - \kappa_1(y)\,t)\,(1 - \kappa_2(y)\,t)} \\ &= \frac{\kappa_1(y)}{1 - \kappa_1(y)\,t} + \frac{\kappa_2(y)}{1 - \kappa_2(y)\,t}\,. \end{aligned} \tag{6.88}$$

If $\kappa_1(y) \leq 0$, then (6.88) is immediate. If $0 < \kappa_1(y)$, we use the fact that $t < 1/\bar{\kappa}$ and hence

$$2\,t\,\kappa_1(y)\,\kappa_2(y) \leq t\,(\kappa_1^2(y) + \kappa_2^2(y)) < \kappa_1(y) + \kappa_2(y)\,. \tag{6.89}$$

By (6.87), there is a tubular neighborhood $\Omega_\delta = \{\rho < \delta\} \cap \Omega$ such that each $x \in \Omega_\delta$ has a unique closest point $x' \in \partial\Omega$. In other words, the normal exponential map is a diffeomorphism onto Ω_δ. We can therefore use the

Riccati equation to compute the Hessian of the distance function to $\partial\Omega$ on Ω_δ (this is done on page 355 of [**GiTr**]). Given $x \in \Omega$ with

$$\operatorname{dist}(\partial\Omega, x) = t < \delta\,,$$

then $|\nabla \rho| = 1$ ($\nabla\rho$ is the unit normal to the level set $\{\rho = t\}$). If $x' \in \partial\Omega$ is the closest point to x, then the eigenvalues of the Hessian of ρ at x are given by

$$\frac{-\kappa_1(x')}{1-\kappa_1(x')\,t}\,, \quad \frac{-\kappa_2(x')}{1-\kappa_2(x')\,t}\,, \quad 0\,. \tag{6.90}$$

Given any C^2 function $f : \mathbb{R} \to \mathbb{R}$, the Hessian of $f \circ \rho$ is

$$(f(\rho))_{x_i\,x_j} = f''(\rho)\,\rho_{x_i}\,\rho_{x_j} + f'(\rho)\,\rho_{x_i\,x_j}\,. \tag{6.91}$$

If $f(t) = 2\,a\,t - t^2$ for some constant $a > 0$, so that $f \circ \rho = 2\,a\,\rho - \rho^2$, then $f'(t) = 2\,a - 2\,t$ and $f''(t) = -2$. Combining (6.90) and (6.91), if $x' \in \partial\Omega$ is the closest point to x, then the eigenvalues of the Hessian of $2\,a\,\rho - \rho^2$ at x are given by

$$-\,2(a-t)\,\frac{\kappa_1(x')}{1-\kappa_1(x')\,t}\,, \quad -\,2(a-t)\,\frac{\kappa_2(x')}{1-\kappa_2(x')\,t}\,, \quad -2\,. \tag{6.92}$$

Lemma 6.26. *Let Ω be a bounded mean convex region with smooth boundary, and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is continuous, almost conformal, and harmonic. If $u(\partial D) \subset \partial\Omega$ and $u(D) \cap \Omega \neq \emptyset$, then $u(D) \subset \Omega$ and $|\nabla u|$ does not vanish on ∂D.*

Notice that this implies that $u(\bar{D})$ intersects the boundary transversely.

Proof. Choose $a > 0$ small enough so that

$$a\,\bar{\kappa} < \frac{1}{2}\,; \tag{6.93}$$

then for $t < \delta_1 = \min\{\delta, \frac{1}{2\,\bar{\kappa}}\}$ we have

$$(a-t)\,\frac{\bar{\kappa}}{1-\bar{\kappa}\,t} < 1\,. \tag{6.94}$$

Set

$$f(t) = 2\,a\,t - t^2 \tag{6.95}$$

and

$$v(z) = f \circ \rho(u(z))\,. \tag{6.96}$$

Combining (6.92) and (6.94), we get that the eigenvalues of $\operatorname{Hess}_{f\circ\rho}(\bar{y})$ for $\rho(y) \leq \delta_1$ are

$$-2 < -\,2(a-t)\,\frac{\kappa_2(y)}{1-\kappa_2(y)\,t} \leq -\,2(a-t)\,\frac{\kappa_1(y)}{1-\kappa_1(y)\,t}\,, \tag{6.97}$$

where $y \in \partial\Omega$ is the closest point to $\bar{y}$.

We will show that $v(z)$ is superharmonic if $u(z)$ is in some tubular neighborhood of $\partial\Omega$. Using this, it will again follow that $u(D) \subset \Omega$ and that the normal derivative of u does not vanish at the boundary.

Let $z = x_1 + i\,x_2$ be coordinates for D and y_1, y_2, y_3 be coordinates for $\mathbb{R}^3$. Using the chain rule and the fact that $\Delta u = 0$, we get

$$\begin{aligned} \Delta v(z) = v_{x_i\,x_i}(z) &= (f \circ \rho)_{y_j\,y_k}(u(z))\,(u_j)_{x_i}(z)\,(u_k)_{x_i}(z) \\ &\quad + (f \circ \rho)_{y_j}(u(z))\,(u_j)_{x_i\,x_i}(z) \\ &= (f \circ \rho)_{y_j\,y_k}(u(z))\,(u_j)_{x_i}(z)\,(u_k)_{x_i}(z)\,. \end{aligned} \tag{6.98}$$

If we let Π_z denote the two-plane spanned by $u_{x_1}(z)$ and $u_{x_2}(z)$, then (6.98) together with the fact that u is almost conformal gives

$$\begin{aligned} \Delta v(z) &= \mathrm{Hess}_{f\circ\rho}(u(z))\,(u_{x_i}(z), u_{x_i}(z)) \\ &= |u_{x_1}(z)|^2\ \mathrm{Tr}\,\mathrm{Hess}_{f\circ\rho}(u(z))\big|_{\Pi_z}\,. \end{aligned} \tag{6.99}$$

Suppose now that $\rho(u(z)) = t \leq \delta_1$ so that (6.97) applies and hence

$$\begin{aligned} &\max_{\text{two-planes }\Pi} \mathrm{Tr}\,\mathrm{Hess}_{f\circ\rho}(u(z))\big|_{\Pi} \\ &\qquad = -2(a-t)\left(\frac{\kappa_1(y)}{1-\kappa_1(y)\,t} + \frac{\kappa_2(y)}{1-\kappa_2(y)\,t}\right) \leq 0\,, \end{aligned} \tag{6.100}$$

where the last inequality follows from (6.88) and y is the closest point in $\partial\Omega$ to $u(z)$.

Combining (6.99) and (6.100), v is superharmonic so long as

$$\rho(u(z)) \leq \min\{\delta_1, a\}\,.$$

Since v is nonnegative, vanishes on the boundary, and $u(D) \cap \Omega \neq \emptyset$, it must be strictly positive on D by the maximum principle. By the definition of v this implies that $u(D) \subset \Omega$. Finally, since v achieves its minimum on every point of the boundary, the Hopf boundary point lemma implies that the normal derivative $\frac{dv}{dn}$ is strictly negative on the boundary. Combining this with the chain rule, we get that in the weak sense

$$0 < \frac{dv}{dn} = (f \circ \rho)_{y_j}(u)\,\frac{du_j}{dn} = 2\,a\left\langle \nabla\rho(u)\,,\,\frac{du}{dn}\right\rangle, \tag{6.101}$$

and hence ∇u does not vanish on the boundary. □

As a consequence, we see that the restriction of u to the boundary is an embedding and u can have at most finitely many interior branch points.

Corollary 6.27. *Let Ω be a bounded mean convex region with $\partial\Omega$ smooth, and suppose that $u : \bar{D} \to \bar{\Omega} \subset \mathbb{R}^3$ is C^1 on $\bar{D}$, almost conformal, and harmonic. Let $\mathcal{B}_0$ denote the branch points of u and define $\mathcal{B} = u^{-1}(u(\mathcal{B}_0))$.*

If $u(\partial D) \subset \partial\Omega$ and $u(D) \cap \Omega \neq \emptyset$, then $\mathcal{B}$ is finite and there is a compact set $K \subset D$ such that $\mathcal{B} \subset K$. In particular, the restriction of u to ∂D is an embedding.

Proof. The previous lemma (see (6.101)) gives that $|\nabla u| > 0$ on ∂D and hence, since u is almost conformal, there are no branch points on the boundary of D. Since $u \in C^1(\bar{D})$, this implies that ∇u does not vanish on a neighborhood of the boundary (so that there are no branch points in a neighborhood of the boundary). Therefore, we have that the branch points are compactly contained in D. Since the local description (6.80) implies that they are isolated there can be only finitely many. Let

$$\mathcal{B}_0 = \{z_1, \ldots, z_k\} \tag{6.102}$$

denote the branch points and let

$$\mathcal{B} = u^{-1}(u(\{z_1, \ldots, z_k\}))\,. \tag{6.103}$$

The previous lemma implies that $u^{-1}(u(z_j)) \subset D$ for each j. Since u is continuous, this implies that $u^{-1}(u(z_j))$ is compactly contained in D. Therefore, Lemma 6.24 implies that $u^{-1}(u(z_j))$ is a finite set. Doing this for each $j = 1, \ldots, k$, we get that $\mathcal{B}$ is a finite set of points which is compactly contained in D. □

We are now prepared to prove the main theorem of this section.

Proof of Theorem 6.23. We will suppose that 0 is a false branch point for u of order $d \geq 2$ and that $u(D)$ is not contained in a plane and deduce a contradiction.

We begin by showing that there is some two-plane Π through $u(0)$ such that Π intersects $u(D)$ transversely and

$$\Pi \cap u(\mathcal{B}) = \{u(0)\}\,. \tag{6.104}$$

Let $\tilde{\Pi}$ denote the set of two-planes in $\mathbb{R}^3$ through $u(0)$ such that (6.104) holds.

Since $u(\mathcal{B})$ is a finite set of points, there is some finite set of lines through $u(0)$ which intersect $u(\mathcal{B}) \setminus \{u(0)\}$. Each of these lines is contained in a one-parameter family of two-planes through $u(0)$ so that $\tilde{\Pi}$ is an open set with full measure.

Define the map $P : \mathbf{S}^1 \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\begin{aligned} P(\theta_1, \theta_2, (a_1, a_2)) = u(0) &+ a_1 \,(-\sin\theta_1, \cos\theta_1, 0) \\ &+ a_2 \,(\cos\theta_1 \,\cos\theta_2, \sin\theta_1 \,\cos\theta_2, -\sin\theta_2) \end{aligned} \tag{6.105}$$

so that for each (θ_1, θ_2) we get an affine map whose image is a two-plane through $u(0)$. Since the differential of P is surjective at every point (as a

map to $\mathbb{R}^3$), the parametric version of Sard's theorem applies. Consequently, for any compact immersed submanifold $N \subset \mathbb{R}^3$ the set of (θ_1, θ_2) for which

$$P(\theta_1, \theta_2, \mathbb{R}^2) \tag{6.106}$$

intersects N transversely is of full measure. In particular, we can find a two-plane $\Pi \in \tilde{\Pi}$ which intersects $u(D)$ transversely.

After a rotation of $\mathbb{R}^3$ fixing $u(0)$ and a translation, we may suppose that $u(0) = 0$ and that Π is given by $y_1 = 0$. The fact that Π and $u(D)$ intersect transversely implies that

$$\nabla_{u(D)} y_1 \neq 0\,, \tag{6.107}$$

and hence the nodal set $\{y_1 = 0\} \cap u(D)$ is a collection of compact immersed arcs. Let $\tilde{\gamma}$ denote the connected component of $\{y_1 = 0\} \cap u(D)$ which contains $u(0)$. We can write

$$\tilde{\gamma} = \bigcup_{j=1}^{\ell} \gamma_j\,, \tag{6.108}$$

where each γ_j is a compact immersed curve which is either closed or has two endpoints. Suppose that $u(0) \in \gamma_1$ and hence $0 \in u^{-1}(\gamma_1) \subset \{u_1 = 0\}$.

Moreover, (6.104) and (6.107) imply that $u^{-1}(\gamma_1)$ is a connected curve which is smooth away from 0. Since 0 is a branch point of order $d \geq 2$, we have the description (6.80) for u. Therefore, in a neighborhood of 0, $u^{-1}(\gamma_1)$ is the union of $2d$ disjoint arcs.

Next, we claim that these $2d$ arcs can never intersect again. If any two did, then we would have a bounded nodal domain for the harmonic function u_1. In that case, u_1 would have to be identically zero by the maximum principle implying that $u(D)$ would be contained in the plane $\{y_1 = 0\}$. Therefore these $2d$ arcs stay disjoint to the boundary. Each of these arcs begins at 0 and ends in ∂D (here we used Lemma 6.26 again). The $2d$ endpoints of these curves in ∂D are mapped by u into the 2 endpoints of γ_1. We conclude that two distinct arcs α_1 and α_2 satisfy

$$u(\alpha_i(0)) = u(0) \text{ and } u(\alpha_1(1)) = u(\alpha_2(1))\,, \tag{6.109}$$

where $u(\alpha_1(1)) \in \partial u(D)$. Let σ denote the portion of ∂D between $\alpha_1(1)$ and $\alpha_2(1)$. Since the map u is monotone on the boundary and the curve $u(\partial D)$ is embedded, $u^{-1}(u(\alpha_1(1)))$ is connected and hence $u(\sigma) = u(\alpha_1(1))$. Therefore, the maximum principle implies that u_1 vanishes on the domain bounded by α_1, α_2, and σ. By unique continuation, u_1 vanishes identically on D, giving the desired contradiction. $\square$

5. Embedded Solutions of the Plateau Problem

In this section, we will see that in certain cases the solution to the Plateau problem must be embedded. Following Meeks and Yau, we will see that if the boundary curve is embedded and lies on the boundary of a smooth convex set (and it is null-homotopic in this convex set), then the minimizing solution is embedded. In the previous section, we saw that the solution had to be immersed in this case. Note that some restriction on the boundary curve is certainly necessary to prove such an embeddedness theorem. For instance, if the boundary curve was knotted (for instance, the trefoil), then it could not be spanned by any embedded disk (minimal or otherwise).

Theorem 6.28 (Meeks-Yau, [**MeY1**]). *Let M^3 be a compact Riemannian three-manifold whose boundary is mean convex and let γ be a simple closed curve in ∂M which is null-homotopic in M; then γ is bounded by a least area disk and any such least area disk is properly embedded.*

Theorem 6.28 is contained in their results (see [**MeY1**] and [**MeY2**] and references therein for additional results).

We will consider the special case of the above theorem where M is a compact mean convex region in $\mathbb{R}^3$.

Theorem 6.29 (Meeks-Yau, [**MeY1**]). *Let Ω be a bounded mean convex region in $\mathbb{R}^3$ with smooth boundary and $\Gamma \subset \partial\Omega$ a C^1 simple closed curve. If Γ is null homotopic in Ω, then the solution $u : D \to \Omega$ of the Plateau problem is a proper embedding.*

Note that Rado's theorem, Theorem 6.14, can be viewed as a special case of Theorem 6.29. In this case, the boundary curve is on the boundary of a convex cylinder. In general, a curve is said to be extremal if it lies on the boundary of its convex hull. Prior to the work of Meeks and Yau, embeddedness was known for extremal boundary curves in $\mathbb{R}^3$ with small total curvature by the work of Gulliver and J. Spruck, [**GuSp**]. Subsequently, Almgren and Simon, [**AmSi**], and Tomi and A. J. Tromba, [**ToTr**], proved the existence of some embedded solution for extremal boundary curves in $\mathbb{R}^3$ (but not necessarily for the Douglas-Rado solution produced in Section 3).

By the results of Chapter 4 and the previous sections in this chapter, there exists a solution u to the Plateau problem which is a smooth proper immersion such that its restriction to ∂D is an embedding. For the remainder of this section, u will always be assumed to have these properties.

5.1. Embeddedness near the boundary. The following lemma shows that u is an embedding in a neighborhood of the boundary:

Lemma 6.30. *With u as above there exists some $\delta, \bar{\delta} > 0$ such that $u : D \setminus D_{1-\delta} \to \Omega$ is an embedding and $u(D)$ is embedded in a $\bar{\delta}$ neighborhood of $\partial\Omega$.*

Proof. We will show that $u(w_1) \neq u(w_2)$ for any w_1, w_2 in $D \setminus D_{1-\delta}$ so long as δ is sufficiently small.

Since u is an immersion and is C^1 on $\bar{D}$, there is an $\epsilon_0 > 0$ such that

$$u(D_{\epsilon_0}(z_0) \cap D) \tag{6.110}$$

is an embedded surface for any $z_0 \in D$. Consequently, we may assume that $|w_1 - w_2| > \epsilon_0$.

Since u is C^1 on $\bar{D}$, there is a constant C_0 such that for any $z_1, z_2 \in \bar{D}$

$$|u(z_1) - u(z_2)| \leq C_0 \, |z_1 - z_2| \, . \tag{6.111}$$

The fact that u restricted to ∂D is a C^1 embedding implies that there is some constant $C_1 > 0$ such that for any $z_1, z_2 \in \partial D$,

$$C_1 \, |z_1 - z_2| \leq |u(z_1) - u(z_2)| \, . \tag{6.112}$$

Choose $z_1, z_2 \in \partial D$ with $|w_i - z_i| \leq \delta$. By the triangle inequality together with (6.111) and (6.112),

$$\begin{aligned} |u(w_1) - u(w_2)| &\geq |u(z_1) - u(z_2)| - 2\, C_0\, \delta \geq C_1\, |z_1 - z_2| - 2\, C_0\, \delta \\ &\geq C_1\, (\epsilon_0 - 2\delta) - 2\, C_0\, \delta \, . \end{aligned} \tag{6.113}$$

Equation (6.113) implies that the restriction of u to $D \setminus D_{1-\delta}$ is an embedding so long as $\delta < C_1\, \epsilon_0 / (2(C_0 + C_1))$.

By Lemma 6.26 of the previous section, if

$$v = f \circ \rho \circ u$$

where $f(t) = 2\, a\, t - t^2$ for some $a > 0$ and ρ is the distance to $\partial\Omega$, then v is superharmonic if $v(z) \leq \delta_0$ for some $\delta_0 > 0$ and there exists $\beta > 0$ such that for all $z \in \partial D$,

$$\frac{\partial v}{\partial n}(z) > \beta \, . \tag{6.114}$$

Since u is C^1 on $\bar{D}$, there exists $\delta_3 > 0$ so that (6.114) holds if $|z| > 1 - \delta_3$. Therefore, there exist $\delta_1, \delta_2 > 0$ and $U \subset D$ so that $D_{1-\delta_1} \subset U$ and $v(\partial U) = \delta_2 < \delta_0$. As in the previous section, the maximum principle implies that $v(U) \geq \delta_2$. Therefore, there is some $\bar{\delta} > 0$ such that

$$u^{-1}(\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \bar{\delta}\}) \subset D \setminus D_{1-\delta} \, . \tag{6.115}$$

The second claim now follows from the first. $\square$

5.2. Density and the double points. Recall that for a point $x \in \Omega$ the density of $u(D)$ at x is given by

$$\Theta_x = \lim_{s \to 0} \frac{\mathrm{Vol}(B_s(x) \cap u(D))}{\pi \, s^2} . \tag{6.116}$$

If $x \in u(D)$, then (since $u(D)$ is an immersed submanifold) in any sufficiently small ball, $B_\epsilon(x)$, we have that $B_\epsilon(x) \cap u(D)$ consists of (possibly several) minimal surfaces intersecting at x. By monotonicity, as in Corollaries 1.13 and 1.14, we see that Θ_x is equal to the number of preimages of x. That is,

$$\Theta_x = \left| u^{-1}(x) \right| . \tag{6.117}$$

Clearly, if $x \notin u(D)$, then $\Theta_x = 0$. In either case, (6.117) holds.

Consequently, the set of double points $\mathcal{D}$ is the set of $x \in \Omega$ with

$$\Theta_x \geq 2 . \tag{6.118}$$

By (6.117), $\mathcal{D}$ is the smallest set such that $u(D) \setminus \mathcal{D}$ is embedded. Lemma 6.30 implies that

$$\mathcal{D} \subset \{ x \in \Omega \mid \mathrm{dist}\,(x, \partial\Omega) \geq \bar{\delta} \} . \tag{6.119}$$

Using monotonicity, Proposition 1.12, and Lemma 6.30, (6.117) yields a quantitative form of Lemma 6.24. Namely, there exists

$$C_2 = C_2(\mathrm{Vol}(u(D)), \bar{\delta}) < \infty \tag{6.120}$$

such that each point in $u(D)$ has muliplicity at most C_2.

Lemma 6.31. *With u as above, the set of double points $\mathcal{D} \subset u(D)$ is compact.*

Proof. Lemma 6.30 shows that $\mathcal{D}$ is contained in the interior of $u(D)$ so we can apply monotonicity to $u(D)$ in a neighborhood of each point of $\mathcal{D}$.

By Corollary 1.14, the density is upper semicontinuous and $\mathcal{D}$ is closed. Since it is also bounded, $\mathcal{D}$ is compact. □

5.3. Nontransverse double points. Using the local description of a solution to an elliptic equation, we will show that the set of nontransverse self-intersections, $\tilde{\mathcal{D}} \subset \mathcal{D}$, is a finite set of points. By definition, x is in $\tilde{\mathcal{D}}$ if and only if there are points $z_1 \neq z_2$ in $u^{-1}(x)$ with neighborhoods V_1 of z_1 and V_2 of z_2 such that $u(V_1)$ and $u(V_2)$ intersect tangentially at x. Furthermore, we may suppose that each $u(V_i)$ is a graph of a function ϕ_i over a ball B_ϵ in its tangent plane at x.

We can now apply Lemma 6.17 to get an asymptotic expansion for $\Phi = \phi_1 - \phi_2$ (which is not identically zero by unique continuation). Namely, there exists an integer $n \geq 2$ and an asymptotic expansion

$$\Phi(z) = \operatorname{Re} c\, z^n + \rho(z)\,, \tag{6.121}$$

where $c \neq 0$ and

$$|\rho(z)| + |z|\,|\nabla \rho(z)| \leq C|z|^{n+\alpha}\,, \tag{6.122}$$

for some $\alpha > 0$. It follows immediately from a slight variation of Lemma 6.13 that $u(V_1) \cap u(V_2)$ is homeomorphic to $2n$ arcs meeting at x.

Lemma 6.32. *With u and $\mathcal{D}$ as above, let $\tilde{\mathcal{D}} \subset \mathcal{D}$ denote the set of double points with a nontransverse self-intersection; then $\tilde{\mathcal{D}}$ is a finite set of points.*

Proof. Suppose that $x \in \tilde{\mathcal{D}}$ with

$$u^{-1}(x) = \{z_1, \ldots, z_k\} \subset D\,. \tag{6.123}$$

We will argue for the case in which all of the nontransverse sheets have the same tangent plane. The modifications for the general case are clear.

Since u is a C^1 immersion, there are neighborhoods V_i of z_i such that $u(V_i)$ is graphical over P_i, where P_i is the tangent plane to $u(V_i)$ at x for each $i = 1, \ldots, k$.

Since u is proper, there exists some $\epsilon_0 > 0$ such that

$$u^{-1}(B_{\epsilon_0}(x)) \subset \bigcup_{j=1}^{k} V_j\,. \tag{6.124}$$

By the definition of $\tilde{\mathcal{D}}$, we may suppose that $P_1, \ldots, P_\ell$ all agree up to orientation and there is a definite separation between $\pm P_1$ and $P_{\ell+1}, \ldots, P_k$. Therefore, there is some $\epsilon_1 > 0$ such that for any $j_0 \leq \ell < j_1$, $B_{\epsilon_1}(x) \cap u(V_{j_0})$ and $B_{\epsilon_1}(x) \cap u(V_{j_1})$ intersect transversely.

Finally, the discussion preceding the lemma gives some $\epsilon_{j_0,j_1} > 0$ for each $j_0 \neq j_1 \leq \ell$ so that $B_{\epsilon_{j_0,j_1}}(x) \cap u(V_{j_0})$ and $B_{\epsilon_{j_0,j_1}}(x) \cap u(V_{j_1})$ intersect transversely away from x. If we now let $\epsilon > 0$ be the minimum of ϵ_0, ϵ_1, and all of the ϵ_{j_0,j_1}, then

$$B_\epsilon(x) \cap \tilde{\mathcal{D}} = \{x\} \tag{6.125}$$

and hence $\tilde{\mathcal{D}}$ is an isolated set of points.

It remains to show that $\tilde{\mathcal{D}}$ is compact. To see this, suppose that $x_j \in \tilde{\mathcal{D}}$ and let $z_j^i \in u^{-1}(x_j)$ for $i = 1, 2$ be the points whose neighborhoods intersect tangentially at x_j. Since u is a C^1 immersion, there is an $\epsilon_0 > 0$ such that

$$|z_j^1 - z_j^2| \geq \epsilon_0\,. \tag{6.126}$$

By Lemma 6.30, the z_j^i are compactly contained in Ω and hence have limit points $z^1, z^2 \in D$ which also satisfy (6.126) (and hence $z_1 \neq z_2$).

Since u is C^1, the fact that $u(z_j^1) = u(z_j^2)$ and $|N(z_j^1)| = |N(z_j^1)|$ for each j implies that $u(z^1) = u(z^2)$ and $|N(z^1)| = |N(z^1)|$, and thus $u(z^1) = u(z^2) \in \tilde{\mathcal{D}}$ since $z_1 \neq z_2$. The fact that $z_j^1 \to z_1$ and u is continuous implies that $x_j \to u(z_1)$ and hence $\tilde{\mathcal{D}}$ is compact. □

By the previous lemma, it follows that $\mathcal{D}$ is a collection of compact immersed curves which branch at the finite set of points $\tilde{\mathcal{D}}$.

Corollary 6.33. *With u and $\mathcal{D}$ as above, we can write*

$$\mathcal{D} = \bigcup_{j=1}^{n} \eta_j \,, \tag{6.127}$$

where each η_j is a compact immersed curve.

Proof. The implicit function theorem implies that $\mathcal{D} \setminus \tilde{\mathcal{D}}$ is a union of C^1 immersed curves. Therefore, since Lemma 6.32 gives that $\tilde{\mathcal{D}}$ is a finite set of points and Lemma 6.31 gives that $\mathcal{D}$ is compact, the corollary follows. □

5.4. Folding curves. The final preliminary that we will need is the notion of a folding curve used by Meeks and Yau.

Definition 6.34. Let $f : D_r \to \mathbb{R}^3$ be a Lipschitz map such that the restriction of f to either $x_1 \geq 0$ or $x_1 \leq 0$ is a C^1 immersion up to the boundary. If for each x_2 either of the following two possibilities occurs, then we say that f has a *folding curve* along the x_2-axis:

First, the plane spanned by $f_{x_2}(0, x_2)$ and $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$ is transverse to the plane spanned by $f_{x_2}(0, x_2)$ and $\lim_{x_1 \to 0_-} f_{x_1}(x_1, x_2)$. Second, the vector $\lim_{x_1 \to 0_-} f_{x_1}(x_1, x_2)$ is a negative multiple of $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$.

The main point of folding curves is that they cannot arise in least area maps, as is shown by the following lemma:

Lemma 6.35. *If f is a Lipschitz and piecewise C^1 map from a disk D into $\mathbb{R}^3$ which has a folding curve, then there is a piecewise C^1 Lipschitz variation vector field W which is area decreasing. In particular, f cannot be area-minimizing.*

Proof. Suppose f is least area. Then it has zero mean curvature where it is an immersion. The definition of folding curve allows us to construct an area decreasing variation vector field W along the fold.

Since f is Lipschitz on D and the restriction of f to either $\{x_1 \geq 0\}$ or $\{x_1 \leq 0\}$ is a C^1 immersion up to the boundary, f_{x_2} is well defined

everywhere. On the other hand, the angle between $\lim_{x_1 \to 0_+} f_{x_1}(x_1, x_2)$ and $\lim_{x_1 \to 0_-} -f_{x_1}(x_1, x_2)$ is less than π. Consequently, we can choose a compactly supported Lipschitz vector field W which is piecewise C^1 such that, on the one hand,

$$\lim_{x_1 \to 0_+} \langle W(f(x_1, x_2)), f_{x_1}(x_1, x_2) \rangle > 0, \tag{6.128}$$

but, on the other hand,

$$\lim_{x_1 \to 0_-} \langle W(f(x_1, x_2)), -f_{x_1}(x_1, x_2) \rangle > 0, \tag{6.129}$$

and finally

$$\langle W(f(0, x_2)), f_{x_2} \rangle = 0. \tag{6.130}$$

It follows immediately that, if we take W as the variation vector field, then the first variation of area is negative. □

The analog of Lemma 6.35 in one dimension less is: Given a piecewise C^1 curve γ with an interior discontinuity of the tangent γ', then there is one-parameter deformation which fixes the endpoints but decreases the length. It follows immediately that a minimizing geodesic between two points in a complete Riemannian manifold cannot have such a "fold."

We will first discuss how to use the notion of folding curves to prove the embeddedness result in the special case where $u(D)$ intersects itself transversely. The proof in the general case will combine these ideas with a perturbation argument.

The next result that we need is a purely topological result for self-transverse maps from the disk. Suppose that $v : \overline{D} \to \mathbb{R}^3$ is a C^1 proper immersion (not necessarily minimal) in general position such that the restriction of v to ∂D is an embedding.

We can still define the set of double points in the obvious way (namely points with more than one preimage). Since v is in general position, the image of v intersects itself transversely along a finite union $\mathcal{D}$ of immersed curves. Let $U_1, \ldots, U_n$ denote the connected components of $D \setminus v^{-1}(\mathcal{D})$ so that the restriction of v to each U_j is an embedding.

The disk D is given by the disjoint union of the closures of the U_j together with a series of identifications of the boundaries ∂U_j. These identifications are compatible with v in the sense that v has the same value at any two points which are identified.

The point of Proposition 6.37 below is that there is another way of identifying the boundaries of the ∂U_j to construct D in this manner which is compatible with v and such that v is now a piecewise C^1 embedding which fails to be C^1 at $\mathcal{D}$. After doing this, the new map has self-intersections along

$\mathcal{D}$ but does not cross itself. By doing this, we have introduced folding curves along $\mathcal{D}$. The following example in one dimension less illustrates this:

Example 6.36. Let $v : [-\pi, \pi] \to \mathbb{R}^2$ be given by

$$v(t) = (2\,t/\pi - \sin t, \cos t)\,. \tag{6.131}$$

Then the double set is given by $(0,0)$, its inverse image is $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$, and we have $U_1 = (-\pi, -\frac{\pi}{2})$, $U_2 = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $U_3 = (\frac{\pi}{2}, \pi)$. Initially, $[-\pi, \pi]$ is given by taking the union of the (relative) closures of the U_i and identifying $-\frac{\pi}{2} \in \overline{U_1}$ with $-\frac{\pi}{2} \in \overline{U_2}$ and $\frac{\pi}{2} \in \overline{U_2}$ with $\frac{\pi}{2} \in \overline{U_3}$.

On the other hand, we can also construct the interval $[-\pi, \pi]$ by identifying $-\frac{\pi}{2} \in \overline{U_1}$ with $\frac{\pi}{2} \in \overline{U_2}$ and $-\frac{\pi}{2} \in \overline{U_2}$ with $\frac{\pi}{2} \in \overline{U_3}$. With this identification, v is still continuous since $v(-\frac{\pi}{2}) = v(\frac{\pi}{2})$ (this is what we mean by saying that the identification is compatible with v). By construction, the image of v is the same.

Clearly, v is no longer C^1 across $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. However, the image of v does not cross itself, although it does still have a self-intersection at a singular point (this is what we mean by saying that v is a piecewise C^1 embedding). We can now deform v to an embedding by making an arbitrarily small C^0 deformation in a neighborhood of $(0,0)$. This operation, which is known as "rounding off", will reduce the length of the image of v.

We shall need the following topological proposition. See Freedman, [**Fr**], for the proof which uses the tower construction from topology. This construction goes back to Papakyriakopolous, [**Pa**].

Proposition 6.37. *Suppose that $v : \overline{D} \to \mathbb{R}^3$ is a C^1 proper immersion in general position such that the restriction of v to ∂D is an embedding. If $\mathcal{D}$ is the set of double points as above, let $U_1, \dots, U_n$ denote the connected components of $D \setminus v^{-1}(\mathcal{D})$. By constructing appropriate identifications compatible with v, we can form a new disk $D' = \bigcup_{j=1}^{n} \overline{U}_j$ with the following properties. The map $w : \overline{D'} \to \mathbb{R}^3$ given by $w|_{U_i} = v|_{U_i}$ is a Lipschitz continuous piecewise C^1 embedding. The maps w and v have the same image and agree on $\partial D = \partial D'$. Finally, w is arbitrarily close in C^0 to an embedding, and each component of $\partial U_i \setminus \partial D$ is a folding curve.*

The proof of Theorem 6.29 in the case where $u(D)$ is self-transverse is now clear. Namely, we use Proposition 6.37 to produce a folding curve for a map with the same area and boundary values. Lemma 6.35 implies that this new map is not area-minimizing and, hence, neither was u. This contradiction implies that u must have been an embedding in the first place.

Using ideas of Freedman, [**Fr**], and Freedman, Hass and Scott, [**FHS**], we can obtain the general case by combining the above approach with a perturbation argument.

Lemma 6.38. *With u and $\mathcal{D}$ as above, there exist $\epsilon > 0$ and $x_1 \in \mathcal{D}$ such that $B_\epsilon(x_1) \cap \mathcal{D}$ has only transverse self-intersections.*

Proof. This follows immediately since the nontransverse self-intersections are isolated. □

Before continuing, we will recall the local description of immersed minimal surfaces near points of self-intersection. This result is a direct consequence of the analytic results on nodal and critical sets of elliptic equations (similar to the arguments in Section 2).

Lemma 6.39 (Freedman, Hass, and Scott, [**FHS**]). *Let $U \subset D$ be a neighborhood of 0, u_1, u_2 be smooth functions on U, and suppose that $v = u_1 - u_2$ satisfies $v = c \operatorname{Re} z^d + q(x)$ where $c \neq 0$ and q satisfies (6.39). Let $h : U \to V$ be a C^1 diffeomorphism where $V \subset D$ is a neighborhood of 0 such that $c \operatorname{Re} z^d(h(x)) = v(x)$ for $x \in U$.*

Let $r > 0$ be such that $D_r \subset U$ and ψ be a smooth function on U with support in D_r such that $\psi(x) = 1$ for $|x| \leq \frac{r}{2}$. For $t > 0$ set $u_t(x) = u_1(x) + t\,\psi(x)$; then there exists $t_0 > 0$ such that for $0 < t \leq t_0$ the graphs of u_t and u_2 meet transversely.

Remark 6.40. Notice that since the intersection is not changed above ∂D_r, the new curves of intersection have the same endpoints as the original (singular) curves.

5.5. Embeddedness. We are now prepared to prove the main result of this section.

Proof of Theorem 6.29. Thus far, we have shown that the solution $u : D \to \Omega$ of the Plateau problem is a proper smooth immersion which is an embedding in a neighborhood of the boundary. It remains to show that the set of double points $\mathcal{D}$ is empty. We will assume that $\mathcal{D} \neq \emptyset$ and obtain a contradiction from this.

By Lemma 6.38, there exist $\epsilon_1 > 0$ and $x_1 \in \mathcal{D}$ such that $B_{\epsilon_1}(x_1) \cap \mathcal{D}$ has only transverse self-intersections. It follows from Corollary 6.33 that are only finitely many possible cut-and-paste operations (as in Proposition 6.37) to make $B_{\epsilon_1}(x_1) \cap u(D)$ embedded. By Lemma 6.35, each of these cut-and-paste operations reduces the area of $u(D)$ by at least some $\epsilon_2 > 0$ (by perturbing along the resulting folding curves).

Lemma 6.30 implies that there is some $\delta > 0$ such that $u(D_{1-\delta}) \cap \mathcal{D} = \emptyset$. Consequently, the local description, Lemma 6.39, allows us to perturb the map u in

$$D_{1-\delta} \setminus u^{-1}(B_{\epsilon_1}(x_1)) \tag{6.132}$$

by an arbitrarily small amount to produce a new map $\tilde{u}$ which is in general position. In particular, we may assume that

$$\text{Area}(\tilde{u}(D)) < \text{Area}(u(D)) + \frac{\epsilon_2}{2}\,. \tag{6.133}$$

By Proposition 6.37, we may cut-and-paste D to obtain a new disk D' such that the identifications are compatible with $\tilde{u}$. The piecewise smooth embedding $w : D' \to \Omega$ has the same image, and thus the same area, as $\tilde{u}$ and these maps agree on the boundary. Since these images agree, the identifications made in $u^{-1}(B_{\epsilon_1}(x_1))$ must be among those listed above. In particular, by perturbing along one of the resulting folding curves in $B_{\epsilon_1}(x_1)$ we may reduce the area of $w(D')$ by at least ϵ_2. Together with (6.133) this contradicts the minimality of $\text{Area}(u(D))$, and the proof is complete. □

Chapter 7

Minimal Surfaces in Three-Manifolds

In this chapter, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). In the next section, we prove eigenvalue estimates of Hersch, and Yang and Yau. After that, we turn to an integral formula of Reilly and the Choi-Wang eigenvalue estimate for embedded minimal surfaces. Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that, by the results of Choi and Wang, and Yang and Yau, such minimal surfaces have uniform area bounds. In the next section, we prove the positive mass theorem of Schoen and Yau. Finally, in the last section, we prove the Colding-Minicozzi estimate for the extinction time of Ricci flow on a homotopy sphere.

1. The Minimal Surface Equation in a Three-Manifold

In this section, we will first describe the modifications necessary to discuss minimal surfaces in general Riemannian three-manifolds. The local picture is very similar to that of minimal surfaces in $\mathbb{R}^3$. In fact, many regularity results for minimal surfaces in arbitrary three-manifolds can be reduced to the corresponding global result in $\mathbb{R}^3$ by means of a rescaling argument.

In the following, M^n will denote a complete n-dimensional Riemannian manifold with sectional curvature bounded by k (i.e., $|K_M| \leq k$) and injectivity radius bounded below by $i_0 > 0$. We will usually take $n = 3$.

In order to establish the necessary local results relating curvature and area for minimal surfaces, we shall need to recall some preliminary geometric facts. The standard Hessian comparison theorem implies the following:

Lemma 7.1. *For* $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$ *and any vector* X *with* $|X| = 1$,

$$\left|\mathrm{Hess}_r(X,X) - \frac{1}{r}\langle X - \langle X, Dr\rangle Dr, X - \langle X, Dr\rangle Dr\rangle\right| \leq \sqrt{k}\,. \tag{7.1}$$

Proof. The Hessian of the distance function vanishes in the radial direction. By the Hessian comparison theorem, the remaining eigenvalues of the Hessian are bounded above and below by $\sqrt{k}\coth\sqrt{k}\,r$ and $\sqrt{k}\cot\sqrt{k}\,r$, respectively. Using this, the bound (7.1) follows from elementary inequalities. □

Let $x \in \Sigma^2 \subset M^3$ with Σ minimal. When M is $\mathbb{R}^n$, the minimality of Σ implies that

$$\Delta r^2 = 4\,. \tag{7.2}$$

This fact is the key to proving the monotonicity formula for minimal surfaces, i.e., Proposition 1.12. In the general case, using minimality and (7.1), we get that

$$\left|\Delta r^2 - 4\right| \leq 4\sqrt{k}\,r \tag{7.3}$$

for $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$. Applying the coarea formula (i.e., (1.59)) and using Stokes' theorem gives

$$s\,\frac{d}{ds}\,\mathrm{Area}(B_s \cap \Sigma) = \int_{\partial B_s \cap \Sigma} \frac{r}{|\nabla r|} \geq \frac{1}{2}\int_{B_s \cap \Sigma} \Delta r^2\,. \tag{7.4}$$

Combining (7.3) and (7.4) implies that for $s < \min\{i_0, \frac{1}{\sqrt{k}}\}$,

$$\frac{d}{ds}\left(\mathrm{e}^{2\sqrt{k}\,s}\,s^{-2}\,\mathrm{Area}(B_s \cap \Sigma)\right) \geq 0\,. \tag{7.5}$$

We could argue similarly, as in Proposition 1.15, to obtain a mean value inequality for minimal surfaces in a three-manifold.

When $M = \mathbb{R}^3$, the Gauss map of a minimal surface is conformal. More generally, the Gauss equation and minimality together imply that

$$K_\Sigma = K_M - \frac{1}{2}|A|^2\,, \tag{7.6}$$

so that

$$|A|^2 \leq 2\,k - 2\,K_\Sigma\,. \tag{7.7}$$

In particular, (7.6) implies that the Gauss map is quasi-conformal. A map $F : (M_1, g_1) \to (M_2, g_2)$ is said to be *quasi-conformal* if there exists a constant $\Lambda < \infty$ such that for all $x \in M_1$ the ratio of the maximum and

minimum eigenvalues of F^*g_2 are bounded by Λ. Note that a conformal map is necessarily quasi-conformal (with $\Lambda = 1$).

In the rest of this section, we will work in local coordinates (x_1, x_2, x_3) with a metric g_{ij} on M. Set e_i equal to the vector field $\frac{\partial}{\partial x_i}$ so that $\langle e_i, e_j \rangle = g_{ij}$. We will use Γ_{ij}^n to denote the Christoffel symbols of the corresponding Riemannian connection.

Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u:

$$\Sigma = \mathrm{Graph}_u = \{(x_1, x_2, u(x_1, x_2)) \mid (x_1, x_2) \in \Omega\}\,. \tag{7.8}$$

We will first derive the minimal surface equation for Σ in these coordinates. In order to do this, we need to express the mean curvature of Σ in terms of u.

For $i = 1, 2$ we define vector fields E_i and linear maps T_i^n by

$$E_i = e_i + u_{x_i} e_3 \equiv T_i^n \, e_n\,, \tag{7.9}$$

so that (E_1, E_2) give a basis for the tangent space to Σ. Let $h_{ij} = \langle E_i, E_j \rangle$ denote the induced metric on Σ. It is convenient to define W by

$$W^2 = 1 + g^{ij} u_{x_i} u_{x_j}\,. \tag{7.10}$$

If N denotes the upward pointing unit normal to Σ, then $\langle N, E_i \rangle = 0$. Therefore, for $i = 1, 2$,

$$\langle N, e_i \rangle = -\frac{u_{x_i}}{W} \tag{7.11}$$

and

$$\langle N, e_3 \rangle = \frac{1}{W}\,. \tag{7.12}$$

The mean curvature is given by

$$h^{ij} \langle N, \nabla_{E_i} E_j \rangle\,. \tag{7.13}$$

Using (7.9), we compute that

$$h^{ij} = g^{ij} - W^{-2} g^{in} g^{jl} u_{x_n} u_{x_l} \tag{7.14}$$

and

$$\begin{aligned} \nabla_{E_i} E_j &= T_i^n \nabla_{e_n} T_j^l e_l = T_i^n \, T_j^l \nabla_{e_n} e_l + \left(T_i^n \nabla_{e_n} T_j^l \right) e_l \\ &= T_i^n \, T_j^l \Gamma_{nl}^m e_m + E_i(T_j^l) \, e_l = T_i^n \, T_j^l \Gamma_{nl}^m e_m + u_{x_i x_j} e_3\,. \end{aligned} \tag{7.15}$$

Combining (7.11), (7.12), and (7.15), we get

$$\langle N, \nabla_{E_i} E_j \rangle = -\frac{1}{W} \sum_{m=1}^{2} \left(u_{x_m} T_i^n \, T_j^l \Gamma_{nl}^m \right) + \frac{1}{W} \left(u_{x_i x_j} + T_i^n \, T_j^l \Gamma_{nl}^3 \right)\,. \tag{7.16}$$

Multiplying through by W and h^{ij}, we get the minimal surface equation

$$0 = h^{ij}\left(u_{x_i x_j} + T_i^n\, T_j^l \Gamma_{nl}^3\right) - \sum_{m=1}^{2}\left(u_{x_m} h^{ij} T_i^n\, T_j^l \Gamma_{nl}^m\right). \tag{7.17}$$

Substituting in the definition of $T_i^n = \delta_i^n + u_{x_i}\, \delta_3^n$, we can write

$$\begin{aligned} T_i^n\, T_j^l \Gamma_{nl}^m &= (\delta_i^n + u_{x_i}\, \delta_3^n)\,(\delta_j^l + u_{x_j}\, \delta_3^l)\, \Gamma_{nl}^m \\ &= \Gamma_{ij}^m + u_{x_i} \Gamma_{3j}^m + u_{x_j} \Gamma_{i3}^m + u_{x_i} u_{x_j} \Gamma_{33}^m . \end{aligned} \tag{7.18}$$

Substituting in (7.18) into (7.17),

$$\begin{aligned} 0 = h^{ij}&\left(u_{x_i x_j} + \Gamma_{ij}^3 + u_{x_i} \Gamma_{3j}^3 + u_{x_j} \Gamma_{i3}^3 + u_{x_i} u_{x_j} \Gamma_{33}^3\right) \\ &- \sum_{m=1}^{2} u_{x_m} h^{ij}\left(\Gamma_{ij}^m + u_{x_i} \Gamma_{3j}^m + u_{x_j} \Gamma_{i3}^m + u_{x_i} u_{x_j} \Gamma_{33}^m\right). \end{aligned} \tag{7.19}$$

We next use this to define a function $F : \mathbb{R}^9 \to \mathbb{R}$ given by

$$\begin{aligned} F(x_1, x_2, u, p_1, p_2, p_{ij}) &= h^{ij}\left(p_{ij} + \Gamma_{ij}^3 + p_i \Gamma_{3j}^3 + p_j \Gamma_{i3}^3 + p_i p_j \Gamma_{33}^3\right) \\ &\quad - \sum_{m=1}^{2} p_m h^{ij}\left(\Gamma_{ij}^m + p_i \Gamma_{3j}^m + p_j \Gamma_{i3}^m + p_i p_j \Gamma_{33}^m\right), \end{aligned} \tag{7.20}$$

where $\Gamma_{ij}^n = \Gamma_{ij}^n(x_1, x_2, u)$ and h^{ij} is the inverse matrix to

$$h_{ij} = g_{ij}(x_1, x_2, u) + p_i g_{j3}(x_1, x_2, u) + p_j g_{3i}(x_1, x_2, u) + p_i p_j g_{33}(x_1, x_2, u).$$

By construction, if u is a solution to the minimal surface equation, then

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_i x_j}) = 0. \tag{7.21}$$

If ∇u is bounded, then (7.21) is uniformly elliptic. Thus, if $|\nabla u|$ is bounded, then we get $C^{2,\alpha}$ estimates for u in terms of the maximum of $|u|$.

We will give two further implications of the form of (7.21), first a removable singularities result and then a local description for the intersection of two minimal surfaces.

The following removable singularities result is far from optimal. In fact, results on removable singularities are much stronger for the minimal surface equation than for linear elliptic equations (see, for instance, chapter 10 of [**Os2**]). However, this result will suffice for our later applications.

Lemma 7.2. *Let u be a C^1 function on $B_1 \subset \mathbb{R}^2$ with*

$$|u| + |\nabla u| \leq C \text{ on } B_1 . \tag{7.22}$$

If u is a $C^2(B_1 \setminus \{0\})$ solution to the minimal surface equation, then u is a smooth solution on all of B_1.

Proof. The assumption (7.22) implies that the minimal surface equation is uniformly elliptic with bounded coefficients. Writing this equation in divergence form, the uniform gradient bound (7.22) and the fact that u is a C^2 solution away from 0 imply that u is a $W^{1,2}$ weak solution on all of B_1. Elliptic estimates (theorem 8.8 of [**GiTr**]) imply that u is in $H^2(B_1)$ so that $|\nabla u| \in W^{1,2}(B_1)$. □

Suppose now that u and v are both smooth solutions to (7.21). Let $w = v - u$ and for $0 \leq s \leq 1$ define

$$G(x_1, x_2, s) = F(x_1, x_2, u+sw, (u+sw)_{x_1}, (u+sw)_{x_2}, (u+sw)_{x_i x_j}) . \tag{7.23}$$

By the fundamental theorem of calculus and the chain rule,

$$\begin{aligned} 0 = G(x_1, x_2, 1) - G(x_1, x_2, 0) &= \int_{s=0}^{1} \frac{d}{ds} G(x_1, x_2, s)\, ds \\ &= \int_{s=0}^{1} F_u\, w + F_{p_i}\, w_{x_i} + F_{p_{ij}}\, w_{x_i x_j}\, ds , \end{aligned} \tag{7.24}$$

where each partial derivative of F is evaluated at

$$(x_1, x_2, u + sw, u_{x_1} + sw_{x_1}, u_{x_2} + sw_{x_2}, u_{x_i x_j} + sw_{x_i x_j}) . \tag{7.25}$$

In particular, (7.24) shows that w itself satisfies a partial differential equation where the coefficients are integrals of partial derivatives of F.

This description of the minimal surface equation allows us to apply Bers' theorem, i.e., Theorem 6.12, to decribe the intersections of minimal surfaces in a three-manifold.

Theorem 7.3 (Local Description for the Intersections of Minimal Surfaces). *Suppose that $\Sigma_1^2, \Sigma_2^2 \subset M^3$ are smooth connected immersed minimal surfaces that do not coincide on an open set. Then Σ_1 and Σ_2 intersect transversely except at an isolated set of points $\mathcal{D}$. Given $y \in \mathcal{D}$ there exists an integer $d \geq 2$ and a neighborhood $y \in U$ where the intersection consists of $2d$ embedded arcs meeting at y.*

Proof. We may assume that $y \in \mathcal{D}$ is a point of nontransverse intersection. Choose coordinates such that $y = (0, 0, 0)$. We may suppose that Σ_1 and Σ_2 are graphs of functions u and v over their common tangent plane at $(0, 0, 0)$. We have that u, v, u_{x_i}, v_{x_i} all vanish at $(0, 0)$. Define w to be $u - v$.

First, we show that w satisfies a uniformly elliptic differential equation with smooth coefficients

$$a_{ij} w_{x_i x_j} + b_i w_{x_i} + cw = 0 . \tag{7.26}$$

Since we are interested in local properties, we may assume that u, v, and their derivatives are uniformly bounded. It follows immediately that the

coefficients in (7.24) are smooth and bounded and that a_{ij} is symmetric. In order to check ellipticity, we need to compute

$$(7.27)\quad a_{ij} = \int_{s=0}^{1} F_{p_{ij}}(x_1, x_2, u+sw, (u+sw)_{x_1}, (u+sw)_{x_2}, (u+sw)_{x_i x_j})\, ds\,.$$

By the mean value theorem (of calculus), for each (x_1, x_2) there exists some $0 \leq t \leq 1$ such that if $f = u + tw$, then

$$(7.28)\qquad a_{ij}(x_1, x_2) = F_{p_{ij}}(x_1, x_2, f, f_{x_1}, f_{x_2}, f_{x_i x_j})\,.$$

It is easy to see that $F_{p_{ij}} = h^{ij}$, where h^{ij} is the inverse matrix to

$$(7.29)\qquad h_{ij} = g_{ij} + f_{x_i}\, g_{j3} + f_{x_j} g_{3i} + f_{x_i}\, f_{x_j}\, g_{33}$$

and where

$$g_{nl} = g_{nl}(x_1, x_2, f(x_1, x_2))\,.$$

Clearly, h^{ij} is uniformly elliptic if and only h_{ij} is. Finally, since g_{ij} is uniformly elliptic by definition and we may assume that f_{x_i} is arbitrarily small, we may conclude that a_{ij} is uniformly elliptic in some neighborhood $U \subset \mathbb{R}^2$. Since w satisfies (7.26), Theorem 6.12 implies that either w vanishes identically or there exists a linear transformation T such that after rotating by T we can write

$$(7.30)\qquad w(x_1, x_2) = p_d(x_1, x_2) + q(x_1, x_2)\,,$$

where p_d is a homogeneous harmonic polynomial of degree $d < \infty$ and

$$(7.31)\qquad |q(x)| + |x|\,|\nabla q(x)| + \cdots + |x|^d\,|\nabla^d q(x)| \leq C|x|^{d+1}\,.$$

In particular, this implies that the points of nontransverse intersection are isolated. Finally, it follows immediately from a slight variation of Lemma 6.13 that $u(U) \cap v(U)$ is homeomorphic to $2d$ arcs meeting at x. □

Theorem 7.3 and its variation in the case where Σ_i is branched (cf. Theorem 6.18), can be used to extend the results of Chapter 4 to minimal surfaces in a three-manifold M^3.

As an application of Theorem 7.3, we will prove a result of J. Hass on minimal surfaces in three-manifolds with minimal foliations.

Before we explain this result we will need the definition of a lamination (cf., for instance, chapter 8 of W. P. Thurston [**Th**], D. Gabai [**Ga**], or D. Calegari [**Ci**]).

Definition 7.4 (Lamination). A codimension one *lamination* of M^3 is a collection $\mathcal{L}$ of disjoint smooth connected surfaces (called leaves) such that $\bigcup_{\Lambda \in \mathcal{L}} \Lambda$ is closed. Moreover, for each $x \in M$ there exists an open neighborhood U of x and a local coordinate chart, (U, Φ), with $\Phi(U) \subset \mathbb{R}^3$ such that in these coordinates the leaves in $\mathcal{L}$ pass through the chart in slices of the form $(\mathbb{R}^2 \times \{t\}) \cap \Phi(U)$.

A *foliation* is a lamination for which the union of the leaves is all of M. The foliation is said to be *oriented* if the leaves are oriented. The foliation is *minimal* if all of the leaves are minimal submanifolds.

Theorem 7.5 (J. Hass, [**Ha**]). *If M^3 has an oriented codimension one minimal foliation without compact leaves, then M does not contain an immersed minimal $\mathbf{S}^2$.*

Proof. We will suppose that an immersed minimal sphere $\Sigma \subset M$ exists and deduce a contradiction. Since none of the leaves are compact, Σ is not contained in a leaf. Therefore, Theorem 7.3 implies that if $p \in \Sigma$ with $p \in \Lambda \in \mathcal{L}$ has $T_p\Sigma = T_p\Lambda$, then there is a neighborhood $p \in U$ where they intersect transversely in $U \setminus \{p\}$. In particular, the restriction of the foliation to Σ gives rise to a singular foliation whose vector field vanishes at isolated points (of nontransverse intersection). Applying Theorem 7.3, we see that the index of the zeros of the vector field is negative at each zero. The Poincaré-Hopf formula implies that the sum of these indices is the Euler characteristic of Σ (which is two). This gives a contradiction, and the theorem follows. $\square$

Clearly, variations of this argument can be applied more generally to give similar results. Furthermore, certain topological assumptions on M^3 make it possible to prove that a minimal immersed sphere must exist. Chapters 4 and 5 describe some existence results for branched minimal immersions, while Chapter 6 describes how to rule out branch points in certain cases. Using a completely different approach F. Smith proved in [**Sm**] that $\mathbf{S}^3$ (with an arbitrary metric) always admits an embedded minimal $\mathbf{S}^2$; see also [**CD**]. In these cases, Theorem 7.5 implies that every minimal foliation of M has compact leaves (see [**Ha**] for more in this direction).

Finally, note that minimal foliations are interesting for topological reasons. In 1979, Dennis Sullivan, [**Du**], proved that a codimension one foliation is *taut* if and only if there exists a Riemannian metric that makes each leaf minimal. A codimension-one foliation of a three-manifold is said to be "taut" if there is a single transverse circle intersecting every leaf, where a "transverse circle" is a closed loop that is always transverse to the tangent field of the foliation.

2. Hersch's and Yang and Yau's Theorems

In 1970, J. Hersch proved that round metrics on $\mathbf{S}^2$ maximized the product of the first eigenvalue and the area. The key point was a clever balancing argument that allowed him to use the coordinate functions of a conformal map to the round sphere as test functions in the Rayleigh quotient. This

idea was extended to general Riemann surfaces by P. Yang and S.T. Yau in [**YgYa**].

Theorem 7.6 (Hersch, [**Her**]). *If Σ^2 is a topological $\mathbf{S}^2$ with a Riemannian metric, then*

$$\lambda_1(\Sigma)\ \mathrm{Area}(\Sigma) \leq \lambda_1(\mathbf{S}^2)\ \mathrm{Area}(\mathbf{S}^2) = 8\,\pi\,, \tag{7.32}$$

with equality if and only if the metric is round.

Proof. By the uniformization theorem (cf. Lemma 5.22), there exists a conformal diffeomorphism

$$\Phi : \Sigma \to \mathbf{S}^2 \subset \mathbb{R}^3\,.$$

For each $i = 1, 2, 3$, set $\phi_i = x_i \circ \Phi$. Given $x \in \mathbf{S}^2$, let $\pi_x : \mathbf{S}^2 \setminus \{x\} \to \mathbf{C}$ be the stereographic projection and let

$$\psi_{x,t}(y) = \pi_x^{-1}(t(\pi_x(y)))\,, \tag{7.33}$$

then for each t, x this can be extended to a conformal map on $\mathbf{S}^2$. Define $\Psi : \mathbf{S}^2 \times (0,1) \to G$, where G is the group of conformal transformations of $\mathbf{S}^2$, by $\Psi(x,t) = \psi_{x,1/(1-t)}$. Since $\Psi(x,0) = \mathrm{id}_{\mathbf{S}^2}$ for each $x \in \mathbf{S}^2$, Ψ can be thought of as a continuous map on $B_1(0) = \mathbf{S}^2 \times (0,1)/(x,0) \equiv (y,0)$. Set

$$\mathcal{A}(\Psi(x,t)) = \frac{1}{\mathrm{Area}(\Sigma)} \left(\int_\Sigma x_i \circ \Psi(x,t) \circ \Phi \right)_{i=1,2,3} \tag{7.34}$$

so

$$\mathcal{A} : B_1(0) \to \mathbb{R}^3 \text{ and } \lim_{(y,t)\to(x,1)} \mathcal{A}(\Psi(y,t)) = x\,. \tag{7.35}$$

In particular, $\mathcal{A}$ extends to ∂B_1 to a map with degree one. Therefore, by elementary topology, (after possibly replacing Φ by $\psi \circ \Phi$) we can assume that for each i,

$$\int_\Sigma \phi_i = 0\,; \tag{7.36}$$

that is, each ϕ_i is orthogonal to the constant functions. It follows from the usual variational description of the first eigenvalue that for each i,

$$\int_\Sigma |\nabla \phi_i|^2 \geq \lambda_1(\Sigma) \int_\Sigma \phi_i^2\,. \tag{7.37}$$

Summing over i and using that $\Phi(\Sigma) \subset \mathbf{S}^2$ (so $\sum_{i=1}^3 \phi_i^2 = 1$), we get

$$\sum_{i=1}^3 \int_\Sigma |\nabla \phi_i|^2 \geq \lambda_1(\Sigma)\ \mathrm{Area}(\Sigma)\,. \tag{7.38}$$

Now, obviously, since Φ is conformal (so that it preserves energy) and since each x_i is an eigenfunction on $\mathbf{S}^2 \subset \mathbb{R}^3$ with eigenvalue $\lambda_1(\mathbf{S}^2)$, so we get

$$\int_\Sigma |\nabla \phi_i|^2 = \int_{\mathbf{S}^2} |\nabla x_i|^2 = \lambda_1(\mathbf{S}^2) \int_{\mathbf{S}^2} x_i^2 \,. \tag{7.39}$$

Combining (7.38) with (7.39) we get

$$\begin{aligned} \lambda_1(\mathbf{S}^2)\ \mathrm{Area}(\mathbf{S}^2) &= \sum_{i=1}^3 \lambda_1(\mathbf{S}^2) \int_{\mathbf{S}^2} x_i^2 = \sum_{i=1}^3 \int_{\mathbf{S}^2} |\nabla x_i|^2 \\ &= \sum_{i=1}^3 \int_\Sigma |\nabla \phi_i|^2 \geq \lambda_1(\Sigma)\ \mathrm{Area}(\Sigma)\,. \end{aligned} \tag{7.40}$$

In the case of equality in (7.32), we see that equality holds in (7.37) for each i, so we must have

$$\Delta\, \phi_i + \lambda_1(\Sigma)\, \phi_i = 0\,. \tag{7.41}$$

Using once again that $\sum_{i=1}^3 \phi_i^2 = 1$, we get

$$\begin{aligned} 0 &= \Delta \sum_{i=1}^3 \phi_i^2 = 2 \sum_{i=1}^3 \phi_i\, \Delta\, \phi_i + 2 \sum_{i=1}^3 |\nabla \phi_i|^2 \\ &= -2\, \lambda_1(\Sigma) \sum_{i=1}^3 \phi_i^2 + 2 \sum_{i=1}^3 |\nabla \phi_i|^2 = -2\, \lambda_1(\Sigma) + 2 \sum_{i=1}^3 |\nabla \phi_i|^2 \,. \end{aligned} \tag{7.42}$$

Hence,

$$\lambda_1(\Sigma) = 2 \sum_{i=1}^3 |\nabla \phi_i|^2 \,. \tag{7.43}$$

Now, since Φ is conformal, we see easily from (7.43) that

$$2\, (d\Phi)^\star\, g_{\mathbf{S}^2} = \lambda_1(\Sigma)\, g_\Sigma \,, \tag{7.44}$$

proving the claim. $\square$

Theorem 7.7 (Yang-Yau, [**YgYa**]). *Suppose that Σ is a surface with a Riemannian metric and $\Phi : \Sigma \to \mathbf{S}^2 \subset \mathbb{R}^3$ is a (branched) conformal map with* $\deg(\Phi) > 0$, *then*

$$\lambda_1(\Sigma)\ \mathrm{Area}(\Sigma) \leq \deg(\Phi)\, \lambda_1(\mathbf{S}^2)\ \mathrm{Area}(\mathbf{S}^2) = 8\, \pi\, \deg(\Phi)\,, \tag{7.45}$$

with equality if and only if Σ is isometric to a round sphere.

Proof. For $i = 1, 2, 3$ set $\phi_i = x_i \circ \Phi$. By the argument given in the beginning of the proof of Hersch's theorem and since Φ is nonconstant, we

can assume that (7.36) holds for each i. The only difference from the proof of Hersch's theorem is now that instead of (7.39) we have that

$$\int_\Sigma |\nabla \phi_i|^2 = \deg(\Phi) \int_{\mathbf{S}^2} |\nabla x_i|^2 = \deg(\Phi)\, \lambda_1(\mathbf{S}^2) \int_{\mathbf{S}^2} x_i^2 \,. \tag{7.46}$$

Carrying through this change proves the theorem. □

Lemma 7.8 (Page 124 of [**FK**]). *Let Σ be a closed surface with genus g. There exists a (branched) conformal map $\Phi : \Sigma \to \mathbf{S}^2$ with degree at most*

$$\rho(g) \leq \begin{cases} g/2+1 & \text{if } g \text{ is even },\\ g/2+3/2 & \text{if } g \text{ is odd }. \end{cases} \tag{7.47}$$

Combining Theorem 7.7 with Lemma 7.8 we get:

Theorem 7.9 (Yang-Yau [**YgYa**]). *Let Σ^2 be a surface with genus g and a Riemannian metric, then*

$$\lambda_1(\Sigma)\ \mathrm{Area}(\Sigma) \leq 4\,\pi\,(g + 5/2 - (-1)^g/2) \leq 8\pi(1+g)\,, \tag{7.48}$$

with equality in the first inequality if and only if Σ is a round sphere.

It is known that if Σ is a Riemann surface with constant negative curvature, then

$$\lambda_1(\Sigma)\ \mathrm{Area}(\Sigma) \leq \pi\, g - \pi\,. \tag{7.49}$$

3. The Reilly Formula

In [**Ry**], R. Reilly discovered a useful integral version of the Bochner formula for domains Ω with boundary $\partial\Omega$.

Lemma 7.10 (Reilly, [**Ry**]). *If u is a smooth function on the bounded domain Ω, then*

$$\begin{aligned} &\int_\Omega \left(\left|\nabla^2 u\right|^2 + \mathrm{Ric}(\nabla u, \nabla u) - (\Delta u)^2 \right) \\ &\qquad = \int_{\partial\Omega} \left(A((\nabla u)^T, (\nabla u)^T) - 2\, u_n\, \Delta_{\partial\Omega} u + H\, u_n^2 \right)\,, \end{aligned} \tag{7.50}$$

where u_n is the normal derivative and $H = -A(e_i, e_i)$ is the mean curvature of $\partial\Omega$.

Proof. The starting point is the Bochner formula (Proposition 1.47)

$$\frac{1}{2}\,\Delta|\nabla u|^2 = \left|\nabla^2 u\right|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle\,. \tag{7.51}$$

Combining this with

$$\mathrm{div}\ (\Delta u\, \nabla u) = (\Delta u)^2 + \langle \nabla u, \nabla \Delta u\rangle\,, \tag{7.52}$$

we get that

$$\operatorname{div}\left(\frac{1}{2}\nabla|\nabla u|^2 - \Delta u\,\nabla u\right) = \left|\nabla^2 u\right|^2 + \operatorname{Ric}(\nabla u, \nabla u) - (\Delta u)^2 \,. \tag{7.53}$$

Therefore, we can apply Stokes' theorem to get

$$\begin{aligned}&\int_{\partial\Omega}\langle\left(\frac{1}{2}\nabla|\nabla u|^2 - \Delta u\,\nabla u\right), \partial_n\rangle \\ &\qquad = \int_\Omega \left(\left|\nabla^2 u\right|^2 + \operatorname{Ric}(\nabla u, \nabla u) - (\Delta u)^2\right),\end{aligned} \tag{7.54}$$

where ∂_n is the outward pointing unit normal to $\partial\Omega$.

Let e_i be an orthonormal frame for $\partial\Omega$. Using the symmetry of the Hessian, we have

$$\frac{1}{2}\langle\nabla|\nabla u|^2, \partial_n\rangle = \langle\nabla_{\partial_n}\nabla u, \nabla u\rangle = \langle\nabla_{\nabla u}\nabla u, \partial_n\rangle\,, \tag{7.55}$$

so we compute

$$\begin{aligned}\langle\nabla_{e_i}\nabla u, \partial_n\rangle &= \langle\nabla_{e_i}(\nabla u)^T, \partial_n\rangle + \langle\nabla_{e_i}(\nabla u)^\perp, \partial_n\rangle \\ &= A(e_i, (\nabla u)^T) + e_i u_n\,,\end{aligned} \tag{7.56}$$

where the last equality also used that $\langle\nabla_{e_i}\partial_n, \partial_n\rangle = 0$. It follows that

$$\begin{aligned}\langle\nabla_{\nabla u}\nabla u, \partial_n\rangle &= u_n\langle\nabla_{\partial_n}\nabla u, \partial_n\rangle + A((\nabla u)^T, (\nabla u)^T) \\ &\quad + \langle(\nabla u)^T, (\nabla u_n)^T\rangle\,.\end{aligned} \tag{7.57}$$

Likewise, we compute

$$\begin{aligned}\Delta u &= \langle\nabla_{\partial_n}\nabla u, \partial_n\rangle + \langle\nabla_{e_i}(\nabla u)^T, e_i\rangle + \langle\nabla_{e_i}(\nabla u)^\perp, e_i\rangle \\ &= \langle\nabla_{\partial_n}\nabla u, \partial_n\rangle + \Delta_{\partial\Omega}u - A(e_i, e_i)\,u_n\,.\end{aligned} \tag{7.58}$$

Finally, combining all of this gives an expression for the boundary terms:

$$\begin{aligned}\left(\frac{1}{2}\partial_n|\nabla u|^2 - \Delta u\,u_n\right) &= A((\nabla u)^T, (\nabla u)^T) + \langle(\nabla u)^T, (\nabla u_n)^T\rangle \\ &\quad - u_n\,\Delta_{\partial\Omega}\,u + H\,u_n^2\,,\end{aligned} \tag{7.59}$$

where $H = -A(e_i, e_i)$ is the mean curvature of $\partial\Omega$. The lemma now follows after integrating by parts on the $\langle(\nabla u)^T, (\nabla u_n)^T\rangle$ term on $\partial\Omega$. □

4. Choi and Wang's Lower Bound for λ_1

In this section, we follow Choi and Wang's application of Reilly's formula to prove a lower bound for the first eigenvalue of a closed embedded minimal hypersurface in a manifold with positive Ricci curvature.

Suppose now that $\partial\Omega$ is minimal (i.e., has $H = 0$). Let λ be the first Dirichlet eigenvalue of $\Delta_{\partial\Omega}$ and let f be a corresponding eigenfunction, so that

$$\Delta_{\partial\Omega} f = -\lambda f \text{ and } \int_{\partial\Omega} f^2 = 1 . \tag{7.60}$$

Let u be the solution of the Dirichlet problem on Ω with boundary data f, so that

$$\Delta_\Omega u = 0 \text{ and } u\big|_{\partial\Omega} = f . \tag{7.61}$$

We will use this function u in Lemma 7.10. The solid integral becomes

$$\int_\Omega \left(\left|\nabla^2 u\right|^2 + \operatorname{Ric}(\nabla u, \nabla u) \right) \geq \int_\Omega \operatorname{Ric}(\nabla u, \nabla u) . \tag{7.62}$$

Meanwhile, the boundary integral becomes

$$\begin{aligned} \int_{\partial\Omega} (A((\nabla u)^T, (\nabla u)^T) - 2\, u_n\, \Delta_{\partial\Omega} u) &= \int_{\partial\Omega} (A((\nabla u)^T, (\nabla u)^T) + 2\,\lambda\, u_n\, u) \\ &= \int_{\partial\Omega} A((\nabla u)^T, (\nabla u)^T) + 2\lambda \int_\Omega |\nabla u|^2 . \end{aligned} \tag{7.63}$$

Combining the boundary and interior terms gives

$$2\lambda \int_\Omega |\nabla u|^2 \geq (\min \operatorname{Ric}) \int_\Omega |\nabla u|^2 - \int_{\partial\Omega} A((\nabla u)^T, (\nabla u)^T) . \tag{7.64}$$

To finish the argument, note that the A term switches sign if we replace Ω by its complement. Thus, we get the theorem of Choi and Wang, [**CiWa**]:

Theorem 7.11 (Choi-Wang, [**CiWa**]). *If M is a simply connected closed manifold with $\operatorname{Ric}_M > k$ and $\Sigma \subset M$ is a closed embedded minimal hypersurface, then the first eigenvalue of Σ is at least $\frac{k}{2}$.*

Finally, note that Choi and Schoen, [**CiSc**], used a covering argument to extend Theorem 7.11 to all closed M's with positive Ricci curvature (i.e., they removed the assumption that M is simply connected).

5. Compactness Theorems with A Priori Bounds

In this section, we will prove the following compactness theorem of Choi and Schoen [**CiSc**]:

Theorem 7.12 (Choi-Schoen, [**CiSc**]). *The space of closed embedded minimal surfaces of genus g in a smooth closed three-manifold M^3 with positive Ricci curvature is compact in the smooth topology.*

The proof of this result uses the positivity of the Ricci curvature to obtain an a priori upper bound on the area of an embedded minimal surface in terms of its genus. Namely, by combining Theorem 7.11 with the earlier Theorem 7.9, Choi and Wang obtained the following corollary:

Corollary 7.13 (Choi-Wang, [**CiWa**]). *If M^3 has $\mathrm{Ric}_M \geq \Lambda > 0$ and $\Sigma_g^2 \subset M$ is a closed embedded minimal surface with genus g, then*

$$\text{Area}\,(\Sigma_g) \leq \frac{16\,\pi\,(1+g)}{\Lambda}\,. \tag{7.65}$$

Integrating (7.7) and applying the Gauss-Bonnet formula, we get

$$\int_\Sigma |A|^2 \leq 2\,k\ \text{Area}\,(\Sigma_g) + 8\,\pi\,(g-1) \leq \frac{32\,\pi\,k\,(1+g)}{\Lambda} + 8\,\pi\,(g-1)\,, \tag{7.66}$$

where the second inequality follows from (7.65). The importance of (7.65) and (7.66) is that the area and total curvature of Σ are bounded uniformly in terms of the genus. This is essential in obtaining a smooth compactness theorem. Note that the area bound alone is enough to give compactness in the space of integral varifolds (see Section 1).

The next proposition shows how to use the bounds (7.65) and (7.66) in combination with Theorem 2.2 to obtain a compactness theorem for minimal surfaces (see [**CiSc**] and cf. M. T. Anderson [**An**] and B. White [**Wh1**]).

Proposition 7.14. *Let M^3 be a closed three-manifold and $\Sigma_i \subset M$ a sequence of closed embedded minimal surfaces of genus g with*

$$\text{Area}(\Sigma_i) \leq C_1 \tag{7.67}$$

and

$$\int_{\Sigma_i} |A_{\Sigma_i}|^2 \leq C_2\,. \tag{7.68}$$

There exists a finite set of points $\mathcal{S} \subset M$ and a subsequence $\Sigma_{i'}$ that converges uniformly in the C^ℓ topology (any $\ell < \infty$) on compact subsets of $M \setminus \mathcal{S}$ to a minimal surface $\Sigma \subset M$. The subsequence also converges to Σ in (extrinsic) Hausdorff distance. Σ is smooth in M, has genus at most g, and satisfies (7.67) *and* (7.68).

Proof. We shall give a proof when $\ell = 2$ since this implies the general case (by standard elliptic theory). Within this proof, $\epsilon = \epsilon(M) > 0$ and $r_0 = r_0(M) > 0$ will be from Theorem 2.2. Let r_1 be such that, by (7.5), for $s < t \leq r_1$ and any minimal surface $\Gamma \subset M$, we have

$$\frac{1}{2} s^{-2}\ \text{Area}(B_s \cap \Gamma) \leq t^{-2}\ \text{Area}(B_t \cap \Sigma)\,. \tag{7.69}$$

In order to find the set $\mathcal{S}$, we define measures ν_i by

$$\nu_i(U) = \int_{U\cap\Sigma_i} |A_i|^2 \leq C_2\,, \tag{7.70}$$

where $U \subset M$ and $A_i = A_{\Sigma_i}$. The general compactness theorem for Radon measures, i.e., Theorem 3.2, implies that there is a subsequence ν_{β_i} which converges weakly to a Radon measure ν with

$$\nu(M) \leq C_2\,. \tag{7.71}$$

For ease of notation, replace ν_i with ν_{β_i}. We define the set

$$\mathcal{S} = \{x \in M \mid \nu(x) \geq \epsilon\}\,.$$

It follows immediately from (7.71) that $\mathcal{S}$ contains at most $C_2\,\epsilon^{-1}$ points.

Given any $y \in M \setminus \mathcal{S}$ we have $\nu(y) < \epsilon$. Since ν is a Radon measure and hence Borel regular, there exists some $0 < 10\,s < \min\{r_0, r_1\}$ (depending on y) such that

$$\nu(B_{10\,s}(y)) < \epsilon\,. \tag{7.72}$$

Since the $\nu_i \to \nu$, (7.72) implies that for i sufficiently large

$$\int_{B_{10\,s}(y)\cap\Sigma_i} |A_i|^2 < \epsilon\,. \tag{7.73}$$

This allows us to apply Theorem 2.2 uniformly to each $B_{10\,s}(y) \cap \Sigma_i$. It follows that for i sufficiently large and $z \in B_{5\,s}(y) \cap \Sigma_i$,

$$25\,s^2\,|A_i|^2(z) \leq 1\,. \tag{7.74}$$

By (a slight variation of) Lemma 2.4, (7.74) implies that for each $z \in B_s(y) \cap \Sigma_i$ the connected component of $B_s(z) \cap \Sigma_i$ containing z is a graph over $U_z^i \subset T_z\Sigma_i$ of a function u_i^z with $|\nabla u_i^z| \leq 1$ and $2\,s\,|\nabla^2 u_i^z| \leq 1$. The bounds on $|\nabla u_i^z|$ and $|\nabla^2 u_i^z|$ imply that u_i^z satisfies a uniformly elliptic equation with Lipschitz coefficients (cf. (7.19)). The usual elliptic estimates (see, for instance, corollary 6.3 of [**GiTr**]) give an $\alpha > 0$ and uniform $C^{2,\alpha}$ estimates for u_i^z on $B_{s/2} \subset T_z\Sigma_i$.

By monotonicity, i.e., (7.69), each connected component of $B_s(y) \cap \Sigma_i$ that intersects $B_{s/2}(y)$ has area at least $\frac{\pi}{8}\,s^2$. Let c_y denote the number of these. By monotonicity, (7.69), and (7.67), we get

$$\frac{\pi}{16}\,r_1^2\,c_y \leq C_1\,. \tag{7.75}$$

In particular, the number of connected components of $B_s(y) \cap \Sigma_i$ which intersect $B_{\frac{s}{2}}(y)$ is bounded independently of both i and y.

Since we have a uniform estimate on the number of components together with uniform $C^{2,\alpha}$ estimates on the graphs, the Arzela-Ascoli theorem gives another subsequence η_i which converges uniformly in $B_s(y)$ in the $C^{2,\frac{\alpha}{2}}$

topology. Since we can cover $M \setminus \mathcal{S}$ by countably many balls like this, a diagonal argument finishes off the convergence to a (possibly immersed) surface Σ which is smooth in $M \setminus \mathcal{S}$. This implies also that Σ satisfies (7.67) and (7.68). Furthermore, the uniform $C^{2,\frac{\alpha}{2}}$ estimates imply that Σ satisfies the same differential equation; that is, Σ is minimal.

We may also suppose that Σ_i converge as integral varifolds to a varifold supported in Σ. By the constancy theorem this must be a multiple of Σ (theorem 41.1 of [**Si4**]). This convergence implies that monotonicity, i.e., (7.69), applies to the limit Σ. Combining this with the area bound, for any $y \in M$ and $r < r_1$ we have

$$r^{-2}\,\text{Area}(B_r(y) \cap \Sigma) \leq 2\,C_1\,. \tag{7.76}$$

Monotonicity also implies that the Σ_i converge to Σ in Hausdorff distance. To see this, note that if we have $y \in \Sigma_i$ with $\text{dist}(y, \Sigma) > 2\delta$, then monotonicity implies that $B_\delta(y) \cap \Sigma_i$ has area at least $C'\,\delta^2$. Since varifold convergence implies that the area measures converge, we see that the Σ_i must converge to Σ in Hausdorff distance.

Since each Σ_i was embedded and the convergence is smooth, Σ cannot cross itself. However, the local description of Theorem 7.3 implies then that Σ must be embedded.

It remains to show that $\Sigma \cup \mathcal{S}$ is an embedded minimal surface. In other words, we must show that each $x \in \mathcal{S}$ is a removable singularity of Σ. Let A denote the second fundamental form of Σ, so that $|A|^2$ is an L^1 function on Σ (using (7.68)). The monotone convergence theorem implies that

$$\lim_{r \to 0} \int_{B_r(x) \cap \Sigma} |A|^2 = 0\,. \tag{7.77}$$

Hence, given any $0 < \delta < 1$ there exists some $0 < r_x < r_0$ with

$$\int_{B_{2\,r_x}(x) \cap \Sigma} |A|^2 < \delta\,\epsilon\,. \tag{7.78}$$

Applying Theorem 2.2 to Σ itself, if $r < r_x$ and $z \in B_r(x) \setminus B_{r/2}(x)$, then

$$r^2\,|A|^2(z) \leq 4\,\delta\,. \tag{7.79}$$

For the moment, fix $r < r_x$ and $z_1 \in \partial B_{3\,r/4}(x)$. A slight variation of Lemma 2.4 and (2.54) imply that, for δ sufficiently small, the component of $B_{r/4}(z_1) \cap \Sigma$ containing z is a minimal graph with gradient bounded by $C\,\sqrt{\delta}$ over $T_z\Sigma$. If we repeat this argument for some

$$z_2 \in \partial B_{3\,r/4}(x) \cap \partial B_{r/8}(z_1)$$

in this minimal graph, then we see that the connected component of

$$B_{r/4}(z_2) \cap \Sigma$$

containing z is also a minimal graph with bounded gradient. The area bound (7.76) implies that after iterating this argument around $\partial B_{3r/4}(x)$ approximately C_1 times we must close up. Taking δ sufficiently small (depending only on C_1), we see that the connected component of $\big(B_r(x) \setminus B_{r/2}(x)\big) \cap \Sigma$ containing z_1 is a graph over a fixed tangent plane with gradient bounded by $C\sqrt{\delta}$ and Hessian bounded by $2\sqrt{\delta}\, r^{-1}$ (see Lemma 2.4).

With $\delta > 0$ small and $r_x > 0$ as above, let Σ_x be a component of $B_{r_x}(x) \cap \Sigma$ with $x \in \bar{\Sigma}_x$. The above discussion shows that Σ_x is a minimal graph of a function u over a fixed tangent plane with $|\nabla u| \leq C\sqrt{\delta}$. We will next see that ∇u has a limit at x. To see this, note that for any $\delta_c > 0$ we may argue as in (7.78) and (7.79) to find some $0 < r_c < r_x$ such that for any $r < r_c$ we have for $z \in \partial B_r(x) \cap \Sigma_x$,

$$r^2 \,|\, \mathrm{Hess}_u \,| \leq 4r^2\, |A|^2(z) \leq 16\, \delta_c\,. \tag{7.80}$$

Integrating this around $\partial B_r(x)$, and using the fact that $\partial B_r(x) \cap \Sigma_x$ is graphical with bounded gradient, we see that

$$\sup_{z_1, z_2 \in \partial B_r(x) \cap \Sigma_x} |\nabla u(z_1) - \nabla u(z_2)| \leq 16\, \pi \sqrt{\delta_c}\,. \tag{7.81}$$

It follows immediately that ∇u has a limit at x. In particular, u can be extended to a C^1 solution of the minimal surface equation with uniformly small gradient. We can now apply Lemma 7.2 to conclude that $\Sigma_x \cup \{x\}$ is a smooth minimal surface.

Repeating this for each element of $\mathcal{S}$, we see that $\Sigma \cup \mathcal{S}$ is a smooth minimal surface which is embedded away from $\mathcal{S}$. The local description, i.e., Theorem 7.3, then implies that it is embedded everywhere. This completes the proof. $\square$

We are now prepared to prove Theorem 7.12. Since we have area and total curvature bounds in this case, Proposition 7.14 implies that any sequence of closed embedded minimal surfaces of fixed genus will have a subsequence which converges away from finitely many points to a smooth embedded minimal surface. The main remaining point is to show that the convergence holds even across these points. There are now several ways to do this, but we will use the lower bound on λ_1 as in the original proof of [**CiSc**].

Proof of Theorem 7.12. By Myers' theorem, since $\mathrm{Ric}_M > 0$, M has finite π_1 so that, after passing to a finite cover, we may suppose that M is simply connected. It suffices to show that given any sequence $\Sigma_i \subset M$ of closed embedded minimal surfaces with genus g, there is a subsequence which converges in the $C^{2,\alpha}$ topology for some $\alpha > 0$.

Since $\mathrm{Ric}_M > 0$, (7.65) and (7.66) hold. Hence, we can apply Proposition 7.14. Consequently, there exists a finite set of points $\mathcal{S} \subset M$ and a

subsequence $\Sigma_{i'}$ that converges uniformly in the $C^{2,\alpha}$ topology on compact subsets of $M \setminus \mathcal{S}$ to a minimal surface $\Sigma \subset M$. Σ is smooth and embedded in M, has genus at most g, and satisfies (7.65) and (7.66). Since M is simply connected and Σ is embedded, the normal bundle to Σ is trivial.

It remains to show that the convergence is smooth across the points $\mathcal{S}$. We will see that this is equivalent to showing that the convergence is multiplicity one (this equivalence also follows from Allard's regularity theorem). First, we will use Theorem 7.11 to see that the convergence is multiplicity one.

Given any $\epsilon > 0$, there exists $i_0 = i_0(\epsilon)$ such that for any $i > i_0$ we have that

$$\Sigma_i \setminus \bigcup_{x_j \in \mathcal{S}} B_{\epsilon^2}(x_j) = \bigcup_{n=1}^{m} \Sigma_{i,n}\,, \tag{7.82}$$

where each $\Sigma_{i,n}$ is a disjoint minimal graph with bounded gradient over Σ. The facts that the Σ_i are embedded and that the normal bundle to Σ is trivial imply that we may choose a top graph which we label $\Sigma_{i,1}$.

Consequently, if the convergence is not multiplicity one, then for large i the minimal surface Σ_i consists of large graphical pieces connected in small sets. Using this description, we will construct a test function which is constant except near these small sets. This will be shown to violate the lower bound on λ_1.

Define a Lipschitz function ϕ on Σ_i by

$$\phi(x) = 1 \text{ for } x \in \Sigma_{i,1} \setminus \bigcup_{x_j \in \mathcal{S}} B_\epsilon(x_j)\,,$$

$$\phi(x) = \frac{\log \operatorname{dist}_M(x, x_j) - 2\log\epsilon}{\log\epsilon - 2\log\epsilon} \text{ for } x \in \Sigma_{i,1} \cap B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j)\,,$$

$$\phi(x) = 0 \text{ for } x \in \Sigma_{i,1} \cap \left(\bigcup_{x_j \in \mathcal{S}} B_{\epsilon^2}(x_j) \right),$$

$$\phi(x) = -\frac{\log \operatorname{dist}_M(x, x_j) - 2\log\epsilon}{\log\epsilon - 2\log\epsilon} \text{ for } n > 1\,, x \in \Sigma_{i,n} \cap B_\epsilon(x_j) \setminus B_{\epsilon^2}(x_j)\,,$$

$$\phi(x) = -1 \text{ for } n > 1 \text{ and } x \in \Sigma_{i,n} \setminus \bigcup_{x_j \in \mathcal{S}} B_\epsilon(x_j)\,.$$

Set $c = \int_{\Sigma_i} \phi$ so that $\psi = \phi - c$ has integral zero. Clearly, we have that

$$|\psi| \geq 1 + |c| \geq 1 \tag{7.83}$$

on at least one of the sheets, and hence

$$\lim_{\epsilon\to 0}\int_{\Sigma_i}\psi^2 \geq \mathrm{Area}(\Sigma)\,. \tag{7.84}$$

On the other hand, $\nabla\psi = \nabla\phi$ and hence the coarea formula (i.e., (1.59)) gives

$$\begin{aligned}\int_{\Sigma_i}|\nabla\psi|^2 &\leq \frac{1}{(\log\epsilon)^2}\sum_{x_j\in\mathcal{S}}\int_{B_\epsilon(x_j)\setminus B_{\epsilon^2}(x_j)}\frac{|\nabla r_j|^2}{r_j^2}\\ &= \frac{1}{(\log\epsilon)^2}\sum_{x_j\in\mathcal{S}}\int_{t=\epsilon^2}^{\epsilon}\int_{\{r_j=t\}\cap\Sigma_i}\frac{|\nabla r_j|}{t^2}\,,\end{aligned} \tag{7.85}$$

where $r_j(x) = \mathrm{dist}(x, x_j)$. We can estimate this using Stokes' theorem and monotonicity to get

$$\begin{aligned}\int_{\{r_j=t\}\cap\Sigma_i}\frac{|\nabla r_j|}{t^2} &= \frac{1}{2}\,t^{-3}\int_{B_t(x_j)\cap\Sigma_i}\Delta r_j^2\\ &\leq 3\,t^{-2}\,\mathrm{Area}(B_t(x_j)\cap\Sigma_i) \leq 3\,C_1\,t^{-1}\,.\end{aligned} \tag{7.86}$$

Substituting (7.86) into (7.85) and integrating,

$$\int_{\Sigma_i}|\nabla\psi|^2 \leq 3\,C_1\,|\mathcal{S}|\,\frac{1}{(\log\epsilon)^2}\int_{\epsilon^2}^{\epsilon}\frac{1}{t}\,dt = 3\,C_1\,|\mathcal{S}|\,\frac{1}{\log\epsilon}\,. \tag{7.87}$$

Therefore, taking ϵ sufficiently small, the test function ψ violates Theorem 7.11. This contradiction shows that $m = 1$, that is, the convergence is multiplicity one.

Using this, we will get uniform curvature estimates for the Σ_i in a neighborhood of each x_j. As before, combining this with the Arzela-Ascoli theorem will imply smooth convergence even across $\mathcal{S}$.

Given $x_j \in \mathcal{S}$ and $0 < \epsilon$, since Σ is smooth we may choose an $0 < s$ such that $B_{2s}(x_j)\cap\Sigma$ is a minimal graph with gradient bounded by ϵ. For i sufficiently large we may suppose that

$$(B_{2s}(x_j)\setminus B_s(x_j))\cap\Sigma_i$$

is the graph over Σ of a function u_i with

$$s^{-1}\,|u| + |\nabla u| \leq 2\epsilon\,.$$

Since the $\Sigma_i \to \Sigma$ in Hausdorff distance, for i sufficiently large we also have Σ_i contained in an $\frac{s}{4}$ tubular neighborhood. Note that the fact that the convergence was multiplicity one was used to conclude that this was a single graph. Since Σ_i is contained in the tubular neighborhood, the boundary of

$B_{2s}(x_j) \cap \Sigma_i$ is $\partial B_{2s}(x_j) \cap \Sigma_i$. Using this description as a graph with small gradient and applying Stokes' theorem to $\Delta_{\Sigma_i} r_j^2$, we get

$$\frac{\text{Area}(B_{2s}(x_j) \cap \Sigma_i)}{4\,\pi\,s^2} \leq 1 + C_3\,\epsilon\,. \tag{7.88}$$

By monotonicity, (7.88) implies that for any $z \in B_s(x_j) \cap \Sigma_i$ we have

$$\frac{\text{Area}(B_s(z) \cap \Sigma_i)}{\pi\,s^2} \leq 1 + C_4\,\epsilon\,. \tag{7.89}$$

Finally, the smooth version of Allard's regularity theorem, i.e., Theorem 2.15, yields uniform curvature estimates for $\epsilon > 0$ sufficiently small, and the proof is complete. □

6. The Positive Mass Theorem

In this section, we will prove the positive mass theorem of Schoen and Yau, [**ScYa1**]. The proof, which roughly follows [**ScYa1**], is an application of minimal surface theory to general relativity.

Let M be asymptotically flat with $\text{Scal}_{M^3} \geq 0$. We assume that M has only one end; the modifications for the general case are straightforward. In particular, there is a compact set $\Omega \subset M$ with $M \setminus \Omega$ is $\mathbb{R}^3 \setminus B_R(0)$ and the metric on $M \setminus \Omega$ can be written as

$$g_{ij} = \left(1 + \frac{\mathcal{M}}{2\,|x|}\right)^4 \delta_{ij} + p_{ij}\,, \tag{7.90}$$

where

$$|x|^2\,|p_{ij}| + |x|^3\,|D\,p_{ij}| + |x|^4\,|D^2\,p_{ij}| \leq C\,. \tag{7.91}$$

$\mathcal{M}$ is the *mass* of M. When $p_{ij} \equiv 0$, $M \setminus \Omega$ is isometric to a constant-time (space-like) slice in the Schwarzschild space-time of mass $\mathcal{M}$.

A tensor h is said to be $O(|x|^{-p})$ if

$$|x|^p\,|h| + |x|^{p+1}\,|D\,h| \leq C\,.$$

For example,

$$\begin{aligned} g_{ij} &= (1 + 2\,\mathcal{M}/|x|)\;\delta_{ij} + O(|x|^{-2})\,, \\ \sqrt{g} \equiv \sqrt{\det g_{ij}} &= 1 + 3\,\mathcal{M}\,|x|^{-1} + O(|x|^{-2})\,. \end{aligned} \tag{7.92}$$

We can now state the positive mass theorem of Schoen and Yau:

Theorem 7.15 (Positive mass theorem; [**ScYa1**]). *If M is as above, then $\mathcal{M} \geq 0$.*

Theorem 7.16 (Uniqueness; [**ScYa1**]). *If $|\nabla^3 p_{ij}| = O(|x|^{-5})$ and $\mathcal{M} = 0$ in 7.15, then $M = \mathbb{R}^3$.*

We have stated only the 3-dimensional version, but Schoen and Yau proved the analogous results through dimension seven; the restriction on dimension comes from the regularity theory of area-minimizing hypersurfaces which allows codimension seven singularities. Witten, [**Wi**], gave a later proof that $\mathcal{M} \geq 0$ using spinors that worked in all dimensions, but only when M has a spin structure (a topological condition on M).

Finally, we note that Penrose conjectured a stronger inequality when M admits a closed minimal surface, where the mass $\mathcal{M}$ is bounded below by $\sqrt{\frac{\mathcal{A}}{16\pi}}$ where $\mathcal{A}$ is the area of the outer-most closed minimal surface. This Penrose inequality was proven by T. Ilmanen and G. Huiskin, [**HuI**], using inverse mean curvature flow. Shortly after, H. Bray, [**Br**], gave a very different proof using the positive mass theorem and giving a stronger estimate when the outer-most minimal surface had multiple components.

6.1. The proof of the positive mass theorem. We will prove Theorem 7.15 by showing that $\mathcal{M} < 0$ leads to a contradiction.

We begin with some calculations. Recall that the Christoffel symbol Γ^i_{jk} is defined by

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \,, \tag{7.93}$$

so that the Hessian of a function f is given by

$$\nabla_{ij} f = f_{ij} - \Gamma^k_{ij} f_k \,. \tag{7.94}$$

Using (7.90), (7.91) and $\partial_i |x| = x_i/|x|$, we compute that

$$\begin{aligned} \Gamma^k_{ij} &= (1/2) g^{km} \left(g_{im,j} + g_{jm,i} - g_{ij,m} \right) \\ &= -\mathcal{M}(x_j \, \delta_{ik} + x_i \, \delta_{jk} - x_k \, \delta_{ij}) \, |x|^{-3} + O(|x|^{-3}) \,. \end{aligned} \tag{7.95}$$

We conclude that, for such a metric,

$$|x|^2 \, |\Gamma^i_{jk}| + |x|^3 \, |R^i_{jkl}| \leq C \,. \tag{7.96}$$

Lemma 7.17. *If $\mathcal{M} < 0$, $\mathrm{Scal}_M \geq 0$, and M asymptotically flat, we can replace g with a new asymptotically flat metric with $\mathrm{Scal}_M \geq 0$, $\mathrm{Scal}_M > 0$ for $|x|$ large, and mass $\mathcal{M}/2$.*

Sketch of proof. By an easy calculation, $\Delta |x|^{-1} < 0$ for $|x|$ large. This allows us to "round off" and obtain $\phi > 0$ with $\Delta \phi \geq 0$, $\Delta \phi > 0$ for $|x|$ large, and ϕ asymptotic to $1 - \mathcal{M}/(4\,|x|)$. Replace g with $\phi^4 \, g$. □

Lemma 7.18. *If $\mathcal{M} < 0$ and M is asymptotically flat, then there exist $R_0, h > 0$ so that for $r > R_0$ the sets*

$$C_r = \{ |x|^2 \leq r^2 \,, \, -h \leq x_3 \leq h \} \tag{7.97}$$

have strictly mean convex boundary.

Proof. Using (7.94) and (7.95), we compute that

$$\nabla_{ij}|x|^2 = 2\delta_{ij} - 2x_k\,\Gamma_{ij}^k = 2\delta_{ij} + O(|x|^{-1})\,. \tag{7.98}$$

Hence, $|x|^2$ is strictly convex for $r > R_0$; in particular, $\{|x|^2 = r^2\}$ is strictly convex for $r > R_0$. Let ν be the unit normal to the slices $\{x_3 = t\}$; we have

$$\nu = (1 - 2\mathcal{M}/|x|)\partial_3 + O(|x|^{-2})\,. \tag{7.99}$$

Since $\sqrt{g} = 1 + 3\mathcal{M}/|x| + O(|x|^{-2})$ and $|x|_i = x_i/|x|$, we compute that

$$\begin{aligned} \operatorname{div}(\nu) &= \sqrt{g}^{-1}(\nu^i\sqrt{g})_i = (1 + O(|x|^{-1}))\left((1 - 2\mathcal{M}/|x|)\delta_{i3} + O(|x|^{-2})\right)_i \\ &= -2\,\mathcal{M}\,x_3\,|x|^{-3} + O(|x|^{-3})\,. \end{aligned} \tag{7.100}$$

Using now that $\mathcal{M} < 0$, there exists $h > 0$ so that the "sandwich"

$$\{-h \le x_3 \le h\} \tag{7.101}$$

has mean convex boundary. □

Proof of Theorem 7.15. By Lemma 7.17, we can assume that $\mathcal{M} < 0$, $\mathrm{Scal}_M \ge 0$, and $\mathrm{Scal}_M > 0$ for $|x| > R$. We will construct a complete asymptotically flat stable minimal surface Σ as a limit of solutions to Plateau problems. Let h, R_0 be from Lemma 7.18 and fix a curve

$$\sigma \subset \{-h \le x_3 \le h\} \setminus B_R \tag{7.102}$$

from $\{x_3 = -h\}$ to $\{x_3 = h\}$. For $s > R_0$, set

$$\gamma_s = \{x_1^2 + x_2^2 = s^2\,,\, x_3 = 0\}\,. \tag{7.103}$$

γ_s has a nonzero linking number with σ.

Solve the Plateau problem (see Chapters 4 and 6) to get area-minimizing embedded minimal disks

$$\Sigma_s \subset C_s\,, \tag{7.104}$$

with $\partial\Sigma_s = \gamma_s$. By a linking argument,

$$\Sigma_s \cap \sigma \neq \emptyset\,.$$

Since the Σ_s's are area minimizing, the standard comparison argument gives uniform density bounds: If $y \in \Omega_s$, $t > R + 2\,h + |y|$, and $B_t(y) \subset C_s$, then

$$\begin{aligned} \mathrm{Area}(B_t(y) \cap \Sigma_s) &\le \mathrm{Area}(C_s \cap \partial B_t(y)) \\ &\quad + \max\{\mathrm{Area}(\{x_3 = h\} \cap B_t(y))\,,\, \mathrm{Area}(\{x_3 = -h\} \cap B_t(y))\} \\ &\le 4\,h\,\pi\,t + O(t) + \pi\,t^2 + O(t) \le \pi\,t^2 + O(t)\,. \end{aligned} \tag{7.105}$$

Finally, the curvature estimate for stable surfaces (see Chapter 2) gives

$$|x|^2\,|A(x)|^2 \le C\,. \tag{7.106}$$

Since $\Sigma \subset \{|x_3| \leq h\}$, (7.106) implies that (for $|x|$ large) each component of Σ is a graph with small gradient; we will sharpen this next. By (7.94) and (7.95),

$$\nabla_{ij} x_3 = -\Gamma_{ij}^3 = (1 + x_3)\, O(|x|^{-3})\,. \tag{7.107}$$

Since Σ is minimal and $|x_3| \leq h$ on Σ, (7.107) implies that

$$\Delta_\Sigma x_3 = O(|x|^{-3})\,. \tag{7.108}$$

Therefore, combining (7.106) with $|x_3| \leq h$, elliptic theory (e.g., corollary 6.3 of [**GiTr**]) gives

$$|x|\, |\nabla_\Sigma x_3| + |x|^2\, |\nabla_\Sigma^2 x_3| \leq C\, h + O(|x|^{-1})\,. \tag{7.109}$$

For graphs with bounded gradient, the Hessian of x_3 and the second fundamental form are essentially equivalent (they are bounded above and below), so we conclude that

$$|x|^4\, |A(x)|^2 \leq C'\,. \tag{7.110}$$

The estimates (7.105)–(7.110) allow us to extract a convergent subsequence and stable limit Σ which also satisfies (7.105), (7.109), (7.110), and the linking condition $\Sigma \cap \sigma \neq \emptyset$. Since each Σ_s is embedded, so is Σ (by the strong maximum principle). Combining (7.105) and (7.109) yields

$$\int_\Sigma |A|^2 < \infty\,. \tag{7.111}$$

Combining this with (7.96) and the Gauss equation, we also have

$$\int_\Sigma |K_\Sigma| < \infty\,. \tag{7.112}$$

Since Σ has quadratic area growth (i.e., (7.105)), we get (logarithmic) cutoff functions $0 \leq \phi_j \leq 1$ with

$$E(\phi_j) \leq 1/j \to 0 \text{ and } \phi_j = 1 \text{ on } \{|x| \leq j\} \cap \Sigma\,. \tag{7.113}$$

Applying the stability inequality with ϕ_j gives

$$\int_\Sigma (|A|^2/2 + \mathrm{Scal}_M - \mathrm{K}_\Sigma)\phi_j^2 \leq\; E(\phi_j) \leq 1/j\,. \tag{7.114}$$

Using (7.111), (7.112), and letting $j \to \infty$, (7.114) gives

$$0 < \int_\Sigma (|A|^2/2 + \mathrm{Scal}_M) \leq \int_\Sigma \mathrm{K}_\Sigma < \infty\,. \tag{7.115}$$

It remains to deduce a contradiction. The key point is that $\mathcal{M} < 0$ implies that Σ is "asymptotically larger than" $\mathbb{R}^2$, contradicting (7.115). We will use the Gauss-Bonnet theorem to make this precise.

For $|x|$ large, each component of Σ is a graph over $\{x_3 = 0\}$ so the area of Σ in large balls is given by the number of components. By (7.105), there

is only one component; i.e., for $|x|$ large, Σ is a single-valued graph over $\{x_3 = 0\}$ of a function u. By (7.109),

$$|x|^2 \, |\nabla^2 u| + |x| \, |\nabla u| + u = O(1) \, . \tag{7.116}$$

Set $\sigma_s = \{x_1^2 + x_2^2 = s^2\} \cap \Sigma = \{(s \cos\theta, s \sin\theta, u(s \cos\theta, s \sin\theta)\}$. Using (7.96) and (7.116), the geodesic curvature of σ_s (in M) is $s^{-1} + O(s^{-2})$. Combined with (7.110), the geodesic curvature k_g of $\sigma_{s'}$ (in Σ) is also $s^{-1} + O(s^{-2})$ and hence

$$\int_{\sigma_s} k_g = (2\,\pi\, s + O(1))\,(s^{-1} + O(s^{-2})) = 2\,\pi + O(s^{-1}). \tag{7.117}$$

On the other hand, for s large, the Gauss-Bonnet theorem gives that

$$\int_{\sigma_s} k_g = 2\,\pi - \int_{\{|x|<s\}\cap\Sigma} K \, . \tag{7.118}$$

For s sufficiently large, (7.115) and (7.118) contradict (7.117), completing the proof. □

7. Extinction of Ricci Flow

In this section, we show that on any homotopy three-sphere the Ricci flow becomes extinct in finite time. This result is due to Colding-Minicozzi, [**CM19**], and Perelman, [**Pe1**]. The proof here follows [**CM19**]. The idea of the proof is to estimate the rate of change of width (defined in Subsection 5.1) and show that the width becomes zero in finite time, thus giving extinction. Theorem 5.26, which proves the existence of good sweepouts, will play a key role in estimating the rate of change of the width.

Let M^3 be a smooth closed orientable three-manifold and $g(t)$ a one-parameter family of metrics on M evolving by Hamilton's Ricci flow, [**Ha1**], so

$$\partial_t g = -2 \, \mathrm{Ric}_{M_t} \, . \tag{7.119}$$

A three-manifold M is *prime* if it cannot be written as a connected sum in a nontrivial way (a connected sum with $\mathbf{S}^3$ is trivial). M is aspherical if all higher homotopy groups vanish; otherwise, it is nonaspherical. Finally, M is a homotopy three-sphere if it has the same homotopy groups as $\mathbf{S}^3$.

When M is prime and nonaspherical, then it follows by standard topology that $\pi_3(M)$ is nontrivial (see, e.g., [**CM19**]). For such an M, fix a nontrivial homotopy class $\beta \in \Omega$. It follows that the width

$$W(g(t)) = W(\beta, g(t))$$

is positive for each metric $g(t)$. This positivity is the only place where the assumption on the topology of M is used in the theorem below giving an upper bound for the derivative of the width under the Ricci flow. As a

consequence, we get that the solution of the flow becomes extinct in finite time (see paragraph 4.4 of [**Pe1**] for the precise definition of extinction time when surgery occurs).

Theorem 7.19 (Colding-Minicozzi, [**CM19**]). *Let M^3 be a closed orientable prime nonaspherical three-manifold equipped with a metric $g = g(0)$. Under the Ricci flow, the width $W(g(t))$ satisfies*

$$\frac{d}{dt} W(g(t)) \leq -4\pi + \frac{3}{4(t+C)} W(g(t)), \tag{7.120}$$

in the sense of the limsup of forward difference quotients. Hence, $g(t)$ becomes extinct in finite time.

The 4π in (7.120) comes from the Gauss-Bonnet theorem and the $3/4$ comes from the bound on the minimum of the scalar curvature that the evolution equation implies. Both of these constants matter, whereas the constant $C > 0$ depends on the initial metric and the actual value is not important.

To see that (7.120) implies finite extinction time rewrite (7.120) as

$$\frac{d}{dt}\left(W(g(t))\,(t+C)^{-3/4}\right) \leq -4\pi\,(t+C)^{-3/4} \tag{7.121}$$

and integrate to get

$$(T+C)^{-3/4}\,W(g(T)) \leq C^{-3/4}\,W(g(0)) - 16\,\pi\,\left[(T+C)^{1/4} - C^{1/4}\right]. \tag{7.122}$$

Since $W \geq 0$ by definition and the right-hand side of (7.122) would become negative for T sufficiently large, we get the claim.

Theorem 7.19 shows, in particular, that the Ricci flow becomes extinct for any homotopy 3-sphere. In fact, we get as a corollary finite extinction time for the Ricci flow on all three-manifolds without aspherical summands (see 1.5 of [**Pe1**] or section 4 of [**CM19**] for why this easily follows):

Corollary 7.20 (Colding-Minicozzi, [**CM19**]; Perelman, [**Pe1**]). *Let M^3 be a closed orientable three-manifold whose prime decomposition has only nonaspherical factors and is equipped with a metric $g = g(0)$. Under the Ricci flow with surgery, $g(t)$ becomes extinct in finite time.*

Part of Perelman's interest in the question about finite time extinction comes from the following: If one is interested in geometrization of a homotopy 3-sphere (or, more generally, a three-manifold without aspherical summands) and knew that the Ricci flow became extinct in finite time, then one would not need to analyze what happens to the flow as time goes to infinity. Thus, in particular, one would not need collapsing arguments.

One of the key ingredients in the proof of Theorem 7.19 is the existence of a sequence of good sweepouts of M, where each map in the sweepout whose area is close to the width (i.e., the maximal energy of any map in the sweepout) must itself be close to a collection of harmonic maps. This is given by Theorem 5.26.

We will use two notions of closeness and convergence of maps from $\mathbf{S}^2$ into a manifold. The first (and weaker) is the varifold distance d_V defined in (3.36) of Chapter 3. The second is bubble convergence defined in the last section of Chapter 3; Proposition 3.27 shows that bubble convergence implies varifold convergence.

7.1. Upper bounds for the rate of change of width. Throughout this subsection, let M^3 be a smooth closed prime and nonaspherical orientable three-manifold and let $g(t)$ be a one-parameter family of metrics on M evolving by the Ricci flow. We will prove Theorem 7.19 giving the upper bound for the derivative of the width $W(g(t))$ under the Ricci flow. To do this, we need three things.

One is that the evolution equation for the scalar curvature $R = R(t)$ (see page 16 of [**Ha2**])

$$\partial_t R = \Delta R + 2|\operatorname{Ric}|^2 \geq \Delta R + \frac{2}{3} R^2 \,, \tag{7.123}$$

implies by a straightforward maximum principle argument that at time $t > 0$,

$$R(t) \geq \frac{1}{1/[\min R(0)] - 2t/3} = -\frac{3}{2(t+C)} \,. \tag{7.124}$$

The curvature is normalized so that on the unit $\mathbf{S}^3$ the Ricci curvature is 2 and the scalar curvature is 6. In the derivation of (7.124) we implicitly assumed that $\min R(0) < 0$. If this was not the case, then (7.124) trivially holds for any $C > 0$, since, by (7.123), $\min R(t)$ is always nondecreasing. This last remark is also used when surgery occurs. This is because by construction any surgery region has large (positive) scalar curvature.

The second thing that we need in the proof is the observation that if $\{\Sigma_i\}$ is a collection of branched minimal 2-spheres and $f \in W^{1,2}(\mathbf{S}^2, M)$ with $d_V(f, \bigcup_i \Sigma_i) < \epsilon$, then for any smooth quadratic form Q on M we have (the unit normal N_f is defined where $J_f \neq 0$)

$$\begin{aligned} &\left| \int_f [\operatorname{Tr}(Q) - Q(N_f, N_f)] - \sum_i \int_{\Sigma_i} [\operatorname{Tr}(Q) - Q(N_{\Sigma_i}, N_{\Sigma_i})] \right| \\ &\qquad < C\, \epsilon \, \|Q\|_{C^1} \operatorname{Area}(f) \,. \end{aligned} \tag{7.125}$$

The last thing is an upper bound for the rate of change of area of minimal 2-spheres. Suppose that X is a closed surface and $f : X \to M$ is a $W^{1,2}$ map, then using (7.119) an easy calculation gives (cf. pages 38–41 of [**Ha2**])

$$\frac{d}{dt}_{t=0} \operatorname{Area}_{g(t)}(f) = - \int_f [R - \operatorname{Ric}_M(N_f, N_f)] \,. \tag{7.126}$$

If $\Sigma \subset M$ is a closed immersed minimal surface, then

$$\frac{d}{dt}_{t=0} \operatorname{Area}_{g(t)}(\Sigma) = - \int_\Sigma \mathrm{K}_\Sigma - \frac{1}{2} \int_\Sigma [|A|^2 + R] \,. \tag{7.127}$$

Here K_Σ is the (intrinsic) curvature of Σ. To get (7.127) from (7.126), we used that if K_M is the sectional curvature of M on the two-plane tangent to Σ, then the Gauss equations and minimality of Σ give $\mathrm{K}_\Sigma = \mathrm{K}_M - \frac{1}{2}|A|^2$. The next lemma gives the upper bound.

Lemma 7.21. *If $\Sigma \subset M^3$ is a branched minimal immersion of the 2-sphere, then*

$$\frac{d}{dt}_{t=0} \operatorname{Area}_{g(t)}(\Sigma) \leq -4\pi - \frac{\operatorname{Area}_{g(0)}(\Sigma)}{2} \min_M R(0) \,. \tag{7.128}$$

Proof. Let $\{p_i\}$ be the set of branch points of Σ and $b_i > 0$ the order of branching. By (7.127),

$$\frac{d}{dt}_{t=0} \operatorname{Area}_{g(t)}(\Sigma) \leq - \int_\Sigma \mathrm{K}_\Sigma - \frac{1}{2} \int_\Sigma R = -4\pi - 2\pi \sum b_i - \frac{1}{2} \int_\Sigma R \,, \tag{7.129}$$

where the equality used the Gauss-Bonnet theorem with branch points (this equality also follows from the Bochner type formula for harmonic maps between surfaces given on page 10 of [**ScYa2**] and the second displayed equation on page 12 of [**ScYa2**] that accounts for the branch points). Note that branch points only help in the inequality (7.128). □

Using these three things, we can show the upper bound for the rate of change of the width.

Proof of Theorem 7.19. Fix a time τ. Below, $\tilde{C}$ denotes a constant depending only on τ but will be allowed to change from line to line. Let $\gamma^j(\tau)$ be the sequence of sweepouts for the metric $g(\tau)$ given by Theorem 5.26. We will use the sweepout at time τ as a comparison to get an upper bound for the width at times $t > \tau$. The key for this is the following claim: Given $\epsilon > 0$, there exist $\bar{j}$ and $\bar{h} > 0$ so that if $j > \bar{j}$ and $0 < h < \bar{h}$, then

$$\begin{aligned} &\operatorname{Area}_{g(\tau+h)}(\gamma^j_s(\tau)) - \max_{s_0} \operatorname{Area}_{g(\tau)}(\gamma^j_{s_0}(\tau)) \\ &\quad \leq [-4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau + C)} \max_{s_0} \operatorname{Area}_{g(\tau)}(\gamma^j_{s_0}(\tau))]\, h + \tilde{C}\, h^2 \,. \end{aligned} \tag{7.130}$$

To see why (7.130) implies (7.120), use the equivalence of the two definitions of widths to get

$$W(g(\tau+h)) \leq \max_{s\in[0,1]} \operatorname{Area}_{g(\tau+h)}(\gamma_s^j(\tau)), \tag{7.131}$$

and take the limit as $j \to \infty$ (so that[1] $\max_{s_0} \operatorname{Area}_{g(\tau)}(\gamma_{s_0}^j(\tau)) \to W(g(\tau)))$ in (7.130) to get

$$\frac{W(g(\tau+h)) - W(g(\tau))}{h} \leq -4\pi + \tilde{C}\,\epsilon + \frac{3}{4(\tau+C)}\, W(g(\tau)) + \tilde{C}\, h. \tag{7.132}$$

Taking $\epsilon \to 0$ in (7.132) gives (7.120).

It remains to prove (7.130). First, let $\delta > 0$ and $\bar{j}$, depending on ϵ (and on τ), be given by Theorem 5.26. If $j > \bar{j}$ and

$$\operatorname{Area}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta,$$

then let $\bigcup_i \Sigma_{s,i}^j(\tau)$ be the collection of minimal spheres given by Theorem 5.26. Combining (7.126), (7.125) with $Q = \operatorname{Ric}_M$, and Lemma 7.21 gives

$$\begin{aligned} \frac{d}{dt}_{t=\tau} \operatorname{Area}_{g(t)}(\gamma_s^j(\tau)) &\leq \frac{d}{dt}_{t=\tau} \operatorname{Area}_{g(t)}(\bigcup_i \Sigma_{s,i}^j(\tau)) \\ &\quad + \tilde{C}\,\epsilon\, \|\operatorname{Ric}_M\|_{C^1} \operatorname{Area}_{g(\tau)}(\gamma_s^j(\tau)) \\ &\leq -4\pi - \frac{\operatorname{Area}_{g(\tau)}(\gamma_s^j(\tau))}{2} \min_M R(\tau) + \tilde{C}\,\epsilon. \end{aligned} \tag{7.133}$$

Substituting the lower bound (7.124) for $R(\tau)$ gives

$$\frac{d}{dt}_{t=\tau} \operatorname{Area}_{g(t)}(\gamma_s^j(\tau)) \leq -4\pi + \frac{3 \max_{s_0} \operatorname{Area}_{g(\tau)}(\gamma_{s_0}^j(\tau))}{4(\tau+C)} + \tilde{C}\,\epsilon. \tag{7.134}$$

Since the metrics $g(t)$ vary smoothly and every sweepout γ^j has uniformly bounded energy, it is easy to see that $\operatorname{Area}_{g(\tau+h)}(\gamma_s^j(\tau))$ is a smooth function of h with a uniform C^2 bound independent of both j and s near $h = 0$ (cf. (7.126)). In particular, (7.134) and Taylor expansion give $\bar{h} > 0$ (independent of j) so that (7.130) holds for s with

$$\operatorname{Area}_{g(\tau)}(\gamma_s^j(\tau)) > W(g) - \delta.$$

In the remaining case, we have

$$\operatorname{Area}(\gamma_s^j(\tau)) \leq W(g) - \delta$$

so the continuity of $g(t)$ implies that (7.130) automatically holds after possibly shrinking $\bar{h} > 0$. □

[1] This follows by combining three facts. First, $\operatorname{Area}_{g(\tau)}(\gamma_{s_0}^j(\tau)) \leq \operatorname{E}_{g(\tau)}(\gamma_{s_0}^j(\tau))$ by (5.86), second, $\max_{s_0} \operatorname{E}_{g(\tau)}(\gamma_{s_0}^j(\tau)) \to W(g(\tau))$, and, third, $W(g(\tau)) \leq \max_{s_0} \operatorname{Area}_{g(\tau)}(\gamma_{s_0}^j(\tau))$ by the equivalence of the two definitions of width.

7.2. When surgeries occur. There are two main points when there are surgeries. The first is that the scalar curvature is positive in the surgery regions and, thus, the lower bound $R \geq -\frac{3}{2(t+C)}$ for the scalar curvature is still valid in the presence of surgeries. The second is that inequality (7.122) is still valid if there are surgeries. Namely, suppose that M is as in Corollary 7.20 and in addition irreducible. From a surgery one can construct a degree-one map from the presurgery manifold to a component of the post-surgery manifold (throwing away the irrelevant "Geometric" components). With Perelman's surgery method (see [**Pe2**]) the map is $(1+\xi)$-Lipschitz for a function ξ which is described in remark 73.6 of [**KL**]. The function $\xi(t)$ depends on the time t of the presurgery manifold, and vanishes as t approaches the surgery time.

Chapter 8

The Structure of Embedded Minimal Surfaces

We will present some recent results on embedded minimal surfaces in $\mathbb{R}^3$. The first section proves a local result from [**CM4**] which shows that an embedded minimal disk is either graphical or, on a slightly larger scale, contains a double-spiral staircase. The next section states the one-sided curvature estimate from [**CM6**]. In the third section, we describe an application (from [**CM10**]) of the one-sided curvature estimate to prove the Generalized Nitsche Conjecture (proven originally by P. Collin, [**Co**]). After this, we turn to the resolution of the Calabi-Yau conjectures for embedded surfaces from [**CM24**]. Finally, we describe the main structure theorems from [**CM7**] for embedded minimal surfaces with finite genus and several recent uniqueness results that have relied upon the structure theory of [**CM3**]–[**CM7**].

1. Disks that are Double-spiral Staircases

We will describe first the local classification of properly embedded minimal disks that follows from [**CM3**]–[**CM6**]. This is the key step for understanding embedded minimal surfaces with finite genus since these can be decomposed into pieces that are disks or pairs of pants. This decomposition uses the monotonicity of topology from Lemmas 1.5 and 1.11.

There are two classical models for embedded minimal disks. The first is a minimal graph over a simply-connected domain in $\mathbb{R}^2$ (such as the plane itself), while the second is a double spiral staircase like the helicoid. A double

spiral staircase consists of two staircases that spiral around one another so that two people can pass each other without meeting.

In [**CM3**]–[**CM6**], we showed that these are the only possibilities and, in fact, every embedded minimal disk is either a minimal graph or can be approximated by a piece of a rescaled helicoid. It is graph when the curvature is small and is part of a helicoid when the curvature is above a certain threshold.

The results of [**CM3**]–[**CM6**], for disks as well as for other topological types, require only a piece of a minimal surface. In particular, the surfaces may well have boundaries and when, for instance, we say in the next theorem, "Any embedded minimal disk in $\mathbb{R}^3$ is either a graph of a function or part of a double spiral staircase", then we mean that if the surface is contained in a Euclidean ball of radius r_0 and the boundary is contained in the boundary of that ball, then in a concentric Euclidean ball with radius a fixed (small) fraction of r_0, any component of the surface is either a graph of a function or part of a double spiral staircase. That the surfaces are allowed to have boundaries is a major point and makes the results particularly useful. Note also that as the conclusion is for a "fixed fraction of the surface" this is an interior estimate.

The following is the main structure theorem for embedded minimal disks[1]:

Theorem 8.1 (Colding-Minicozzi, [**CM3**]–[**CM6**]). *Any embedded minimal disk in $\mathbb{R}^3$ is either a graph of a function or part of a double spiral staircase. In particular, if for some point the curvature is sufficiently large, then the surface is part of a double spiral staircase (it can be approximated by a piece of a rescaled helicoid). On the other hand, if the curvature is below a certain threshold everywhere, then the surface is a graph of a function.*

The proof of Theorem 8.1 has the following three main steps:

A. Fix an integer N (the "large" of the curvature in what follows will depend on N). If an embedded minimal disk Σ is not a graph (or equivalently if the curvature is large at some point), then it contains an N-valued minimal graph which initially is shown to exist on the scale of $1/\max|A|$. That is, the N-valued graph is initially shown to be defined on an annulus with both inner and outer radius inversely proportional to $\max|A|$.

B. Such a potentially small N-valued graph sitting inside Σ can then be seen to extend as an N-valued graph inside Σ almost all the way to the boundary. That is, the small N-valued graph can be extended to an N-valued graph defined on an annulus where the outer radius of the annulus is proportional

[1]See [**CM3**]–[**CM6**] for the precise statements, as well as proofs.

to R. Here R is the radius of the ball in $\mathbb{R}^3$ that the boundary of Σ is contained in.

C. The N-valued graph not only extends horizontally (i.e., tangent to the initial sheets) but also vertically (i.e., transversally to the sheets). That is, once there are N sheets there are many more and, in fact, the disk Σ consists of two multi-valued graphs glued together along an axis.

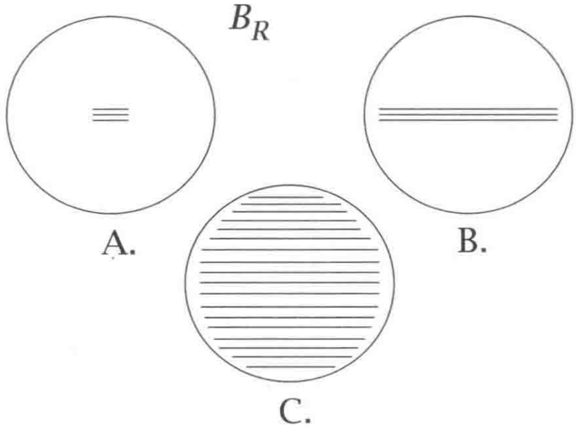

Figure 8.1. Proving Theorem 8.1: A. Finding a small N-valued graph in Σ. B. Extending it in Σ to a large N-valued graph. C. Extend the number of sheets.

The proof is beyond the scope of this book, but we will give the proof of A, relying on some of the material about stable and almost-stable minimal surfaces that we proved in Chapter 2 and results about multi-valued graphs from Chapter 1.

The precise statement of A is that if an embedded minimal disk in a ball has large curvature at a point, then it contains a small almost flat multi-valued graph nearby, that is:

Theorem 8.2 (Colding-Minicozzi, [**CM4**]). *Given $N, \omega > 1$, and $\epsilon > 0$, there exists $C = C(N, \omega, \epsilon) > 0$ so that the following holds:*

Let $0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial \Sigma \subset \partial B_R$. If for some $0 < r_0 < R$ we have

$$\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4\,|A|^2(0) = 4\,C^2\, r_0^{-2}\,,$$

then there exist $\bar{R} < r_0/\omega$ and (after a rotation of $\mathbb{R}^3$) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \setminus D_{\bar{R}}$ with gradient $\leq \epsilon$, and $\operatorname{dist}_\Sigma(0, \Sigma_g) \leq 4\,\bar{R}$.

Recall that by the middle sheet Σ^M of an N-valued graph Σ we mean the portion over

$$\{(\rho, \theta) \in \mathcal{P} \,|\, r_1 < \rho < r_2 \text{ and } 0 \leq \theta \leq 2\,\pi\}\,. \tag{8.1}$$

1.1. Total curvature and area of embedded minimal disks. Using the decomposition of Lemma 2.32 from Chapter 2, we next obtain polynomial bounds for the area and total curvature of intrinsic balls in embedded minimal disks with bounded curvature.

Lemma 8.3. *There exists C_1 so that if $0 \in \Sigma \subset B_{2R}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R}$ and $|A|^2 \leq 4$, then*

$$\int_0^R \int_0^t \int_{B_s^\Sigma} |A|^2 \, ds \, dt = 2(\text{Area}(B_R^\Sigma) - \pi R^2) \leq 6\pi R^2 + 20\, C_1 R^5 . \tag{8.2}$$

Proof. Let the constant C_1, the cutoff function χ, and the set $\bigcup_j \Omega_j$ be given by Lemma 2.32, so that each Ω_j is $\frac{1}{2}$-stable. Define ψ on B_R^Σ by

$$\psi = \psi(\text{dist}_\Sigma(0,\cdot)) = 1 - \text{dist}_\Sigma(0,\cdot)/R ,$$

so $\chi\psi$ vanishes off of the union $\bigcup_j \Omega_j$ of the $\frac{1}{2}$-stable subdomains. Using $\chi\psi$ in the 1/2-stability inequality (see (2.136)), the absorbing inequality and (2.146) give

$$\int |A|^2\chi^2\psi^2 \leq 2\int |\nabla(\chi\psi)|^2 = 2\int \left(\chi^2|\nabla\psi|^2 + 2\chi\,\psi\langle\nabla\chi,\nabla\psi\rangle + \psi^2|\nabla\chi|^2\right)$$

$$\leq 6\, C_1 R^3 + 3\int \chi^2|\nabla\psi|^2 \leq 6\, C_1 R^3 + 3\, R^{-2}\,\text{Area}(B_R^\Sigma) . \tag{8.3}$$

Using (2.145) and $|A|^2 \leq 4$, we get

$$\int |A|^2\psi^2 \leq 4\, C_1 R^3 + \int |A|^2\chi^2\psi^2 \leq 10\, C_1 R^3 + 3\, R^{-2}\ \text{Area}\,(B_R^\Sigma) . \tag{8.4}$$

The lemma follows from (8.4) and Corollary 2.7 that relates area and total curvature for disks. □

The polynomial growth proven in Lemma 8.3 allows us to find large intrinsic balls with a fixed doubling bound on two different scales:

Corollary 8.4. *There exists C_2 so that given $\beta, R_0 > 1$, we get R_2 so that the following holds: .*

If $0 \in \Sigma \subset B_{R_2}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{R_2}$ and

$$\sup_\Sigma |A|^2 \leq 4|A|^2(0) = 4 ,$$

then there exists $R_0 < R < R_2/(2\,\beta)$ with

$$\int_{B_{3R}^\Sigma} |A|^2 + \beta^{-10}\int_{B_{2\beta R}^\Sigma} |A|^2 \leq C_2\, R^{-2}\ \text{Area}\,(B_R^\Sigma) . \tag{8.5}$$

Proof. Set $\mathcal{A}(s) = \text{Area}(B_s^\Sigma)$. Given m, Lemma 8.3 gives

$$(8.6)\qquad \left(\min_{1 \leq n \leq m} \frac{\mathcal{A}((4\beta)^{2n} R_0)}{\mathcal{A}((4\beta)^{2n-2} R_0)} \right)^m \leq \frac{\mathcal{A}((4\beta)^{2m} R_0)}{\mathcal{A}(R_0)} \leq C_1' \, (4\beta)^{10m} \, R_0^3 \, .$$

Fix m with $C_1' \, R_0^3 < 2^m$ and set $R_2 = 2\,(4\beta)^{2m} \, R_0$. By (8.6), there exists $R_1 = (4\beta)^{2n-2} \, R_0$ with $1 \leq n \leq m$ so

$$(8.7)\qquad \frac{\mathcal{A}((4\beta)^2 \, R_1)}{\mathcal{A}(R_1)} \leq 2\,(4\beta)^{10} \, .$$

For simplicity, assume that $\beta = 4^q$ for $q \in \mathbf{Z}^+$. As in (8.6), (8.7), we get $0 \leq j \leq q$ with

$$(8.8)\qquad \frac{\mathcal{A}(4^{j+1} \, R_1)}{\mathcal{A}(4^j \, R_1)} \leq \left[\frac{\mathcal{A}(4\beta \, R_1)}{\mathcal{A}(R_1)} \right]^{1/(q+1)} \leq 2^{\frac{1}{q+1}} \, 4^{10} \, .$$

Set $R = 4^j \, R_1$. Combining (8.7), (8.8), and Corollary 2.7 gives (8.5). □

1.2. Intrinsic sectors. We will show next that sufficiently large "intrinsic sectors" that are almost stable must contain multi-valued graphs. The notion of "almost stable" that we will use is δ-stable as in section 2 of [**CM4**] (see Definition 2.27).

There are two consequences of almost-stability that will be used. The first is the δ-stability inequality (2.136). The second is the following curvature estimate for $1/2$-stable surfaces that follows from the argument of [**CM2**] (cf. Theorem 2.10):

Lemma 8.5. *There exists C so that if $\Sigma \subset \mathbb{R}^3$ is a two-sided $1/2$-stable minimal surface and $B_R^\Sigma(x) \cap \partial\Sigma = \emptyset$, then*

$$|A|^2(x) \leq \frac{C}{R^2} \, .$$

Proof. This follows from the proof of Theorem 2.10 with trivial modifications. See [**CM2**]. □

We will need the notion of an "intrinsic sector" over a subcurve of the boundary of an intrinsic ball. These sectors allow us to subdivide annuli into subsets, similar to subdividing Euclidean annuli in polar coordinates depending on the θ coordinate. Namely, given $\gamma \subset \partial B_r^\Sigma$, define the intrinsic sector; see Figure 8.2,

$$(8.9)\qquad S_R(\gamma) = \{\exp_0(v) \mid r \leq |v| \leq r + R \text{ and } \exp_0(r\,v/|v|) \in \gamma\} \, .$$

Given $R > R_1 > r$, it is convenient to set

$$(8.10)\qquad S_{R_1,R}(\gamma) = S_R(\gamma) \setminus S_{R_1}(\gamma) \, .$$

We will need a few geometric observations about these intrinsic sectors that come from analyzing the Riccati equation:

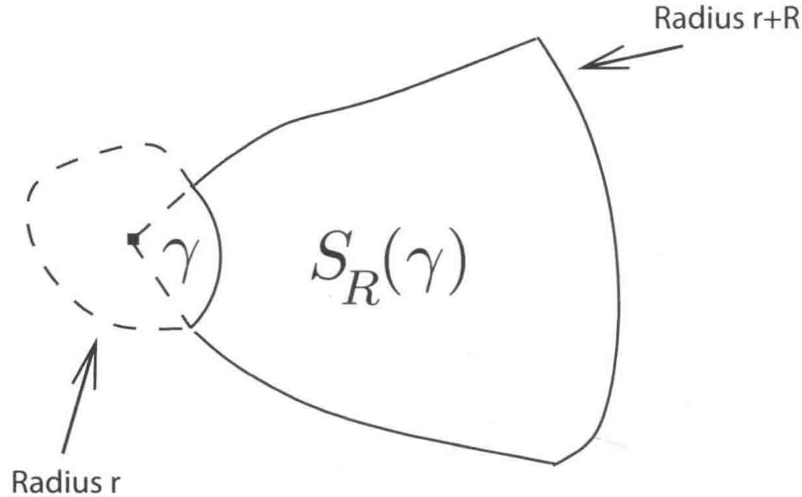

Figure 8.2. The intrinsic sector $S_R(\gamma)$ over a curve $\gamma \subset \partial B_r^\Sigma$ defined in (8.9).

Lemma 8.6. *Suppose that Σ is an embedded minimal disk containing* 0, $\gamma \subset \partial B_r^\Sigma$ *is a compact connected curve, and*

$$B_{r+R}^\Sigma \cap \partial\Sigma = \emptyset\,.$$

Define the normal exponential map

$$\Phi : [0, R] \times \gamma \to S_R(\gamma)$$

by setting $\Phi(\rho, x) = \gamma_x(\rho)$ where γ_x is the unit speed geodesic starting at x and (outward) normal to ∂B_r^Σ. Then:

(1) *Φ is a diffeomorphism and so is $\Phi(\rho, \cdot) : \gamma \to S_R(\gamma) \cap \partial B_{r+\rho}^\Sigma$.*

(2) *If we give the cylinder $[0, R] \times \gamma$ the warped product metric $d\rho^2 + (\rho/r + 1)^2\, dx^2$, then Φ is distance nondecreasing.*

(3) *If k_g is the geodesic curvature of the level sets of ρ, then $k_g \geq \frac{1}{\rho+r}$.*

(4) *If $\sup_{\{\rho \in (3s/4, s)\}} |A|^2 \rho^2 \leq C_A$, then*

$$\sup_{\{\rho = s\}} k_g \leq \frac{C_A + 8}{s}\,.$$

(5) *The integral $\int_{\{\rho=t\}} k_g$ is a monotone nondecreasing function of t.*

Proof. Since Σ is a disk with nonpositive curvature and $B_{r+R}^\Sigma \cap \partial\Sigma = \emptyset$, the (standard) exponential map $\exp_0$ is a diffeomorphism from $B_{r+R} \subset \mathbb{R}^2$ to B_{r+R}^Σ. The Gauss lemma states that the geodesics in Σ starting at 0 are orthogonal to the intrinsic distance spheres centered at 0. This implies that

if $v \in \mathbb{R}^2$ with $r \leq |v| \leq r + R$ and $\exp_0(r\, v/|v|) \in \gamma$, then

$$\exp_0(v) = \Phi(|v| - r, \exp_0(r\, v/|v|))\,. \tag{8.11}$$

It follows that Φ is also a diffeomorphism, giving (1), and standard comparison theorems for nonpositive curvature give (2) and (3).

To see (4), fix a unit-speed geodesic

$$\sigma : [0, r + R] \to \Sigma$$

with $\sigma(0) = 0$ and set $f(t) = k_g \circ \sigma(t)$. Using that the exponential map is a diffeomorphism and Σ is a surface, a standard calculation gives

$$f'(t) = -f^2(t) - K_\Sigma \circ \sigma(t) = -f^2(t) + \frac{1}{2}\, |A|^2 \circ \sigma(t)\,, \tag{8.12}$$

where the last equality used the Gauss equation and minimality. The assumption in (4) gives for $t \in [r + 3s/4, r + s]$ that

$$f'(t) \leq -f^2(t) + \frac{8}{9}\, C_A\, s^{-2} < -f^2(t) + C_A\, s^{-2}\,. \tag{8.13}$$

We now consider two cases:

Case 1: Suppose first that there is some $\bar{t} \in [r + 3s/4, r + s]$ with

$$f(\bar{t}) < \frac{\sqrt{2\, C_A}}{s}\,. \tag{8.14}$$

Since (8.13) implies that $f'(t) \leq C_A\, s^{-2}$ on $[\bar{t}, r + s]$, the fundamental theorem of calculus gives

$$f(r + s) \leq f(\bar{t}) + (r + s - \bar{t})\, \frac{C_A}{s^2} \leq \frac{\sqrt{2\, C_A}}{s} + \frac{C_A}{4s} < \frac{C_A + 1}{s}\,, \tag{8.15}$$

where the last inequality used that $\sqrt{2\, C_A} = 2\, \sqrt{C_A/2} \leq C_A/2 + 1$.

Case 2: Suppose instead that $f(t) \geq \frac{\sqrt{2\, C_A}}{s}$ for all $t \in [r + 3s/4, r + s]$. It follows from (8.13) that, on this interval,

$$f' \leq -f^2 + \frac{C_A}{s^2} \leq -\frac{1}{2}\, f^2\,. \tag{8.16}$$

Since $f(t) \geq \frac{1}{t}$, $1/f$ makes sense and we compute from (8.16) that

$$(1/f)' = -\frac{f'}{f^2} \geq \frac{1}{2}\,, \tag{8.17}$$

so the fundamental theorem of calculus gives that

$$\frac{1}{f(r + s)} \geq \frac{1}{f(r + 3s/4)} + \frac{s}{8} \geq \frac{s}{8}\,. \tag{8.18}$$

We conclude that $f(r + s) \leq \frac{8}{s}$.

Thus, we see that in either case $f(r + s) \leq \frac{C_A + 8}{s}$ and (4) follows.

The last property, (5), follows immediately from the Gauss-Bonnet theorem with corners (see page 274 of [**dC1**]), the Gauss lemma (that gives the angles at the corners), and the nonpositivity of the curvature that comes from the Gauss equation and minimality. □

1.3. Almost stable sectors contain multi-valued graphs. In the next lemma, we use $\mathcal{T}_s(\Omega)$ to denote the s-tubular neighborhood of a set Ω:

$$\mathcal{T}_s(\Omega) = \{x \in \Sigma \,|\, \mathrm{dist}_\Sigma(x, \Omega) < s\}\,.$$

The next lemma is a slight generalization of a result from [**CM3**] that controls the area growth of almost-stable sectors. It is modeled on the results of [**CM2**] (see Chapter 2) for stable disks.

Lemma 8.7 (Colding-Minicozzi, [**CM3**]). *There exists C, so that if $0 \in \Sigma$ is an embedded minimal disk containing a curve $\gamma \subset \partial B^\Sigma_{r_1}$ with*

- $\int_\gamma k_g < C_0\, m$ *and* $\mathrm{Length}(\gamma) \leq 3\,\pi\, m\, r_1$,
- $\mathcal{T}_{r_1/8}(S_{R_1}(\gamma))$ *is 1/2-stable and disjoint from* $\partial\Sigma$,

then for any $t \in (2r_1, 3R_1/4)$ we have

$$\int_\gamma k_g \leq \frac{\mathrm{Length}(\partial B^\Sigma_{r_1+t} \cap S_{R_1}(\gamma))}{t} \leq C\left((C_0+1)m + \frac{R_1}{r_1}\right). \tag{8.19}$$

Furthermore, for any $\Omega > 2$, we get

$$\int_{S_{\Omega r_1, \frac{R_1}{\Omega}}(\gamma)} |A|^2 \leq C\left(\frac{R_1}{r_1} + \frac{(C_0+1)m}{\log \Omega}\right). \tag{8.20}$$

Proof. Within this proof, we will write S_t for the sector $S_t(\gamma)$ to keep notation short. It is also convenient to define a function ρ on S_{R_1} by

$$\rho(x) = \mathrm{dist}_{S_{R_1}}(x, \gamma)\,.$$

By Lemma 8.6, the ρ here agrees with the ρ there.

Define the functions $\ell(t)$ and $K(t)$ by

$$\ell(t) = \mathrm{Length}\left(\{\rho = t\}\right), \tag{8.21}$$

$$K(t) = \int_{S_t} |A|^2\,. \tag{8.22}$$

Since the exponential map is an embedding and the geodesics normal to $\partial\gamma$ are perpendicular to the level sets of ρ (by the Gauss lemma), the first variation formula for arc-length gives

$$\ell'(t) = \int_{\{\rho=t\}} k_g > 0\,. \tag{8.23}$$

Let $d\mu$ be 1-dimensional Hausdorff measure on the level sets of ρ. The Jacobi equation gives

$$\frac{d}{dt}(k_g\, d\mu) = -K_\Sigma\, d\mu = \frac{|A|^2}{2}\, d\mu\,, \tag{8.24}$$

where the second equality uses minimality and the Gauss equation. Define $\bar{K}(t)$ to be the integral of $K(t)$:

$$\bar{K}(t) = \int_0^t K(s)\, ds\,.$$

Integrating (8.24) twice, (8.23) yields

$$\ell(t) = \ell(0) + \int_0^t \left(\int_\gamma k_g + \frac{K(s)}{2} \right) ds = \mathrm{Length}(\gamma) + t \int_\gamma k_g + \frac{\bar{K}(t)}{2}\,. \tag{8.25}$$

Since $\bar{K}(t) \geq 0$, this gives the first inequality in (8.19). Using the coarea formula and $|\nabla \rho = 1|$ and then integrating (8.25) gives

$$\begin{aligned} \frac{\mathrm{Area}(S_{R_1})}{R_1^2} &= \frac{1}{R_1^2} \int_0^{R_1} \ell(t) \leq \frac{\mathrm{Length}(\gamma)}{R_1} + \int_\gamma \frac{k_g}{2} + \frac{1}{R_1^2} \int_0^{R_1} \frac{\bar{K}(t)}{2} \\ &\leq 3\pi\, m + \frac{1}{2} C_0\, m + \frac{1}{R_1^2} \int_0^{R_1} \frac{\bar{K}(t)}{2}\,, \end{aligned} \tag{8.26}$$

where the last inequality used the (assumed) bounds on the length and total curvature of γ.

We will use 1/2-stability to control the total curvature, but this introduces new terms that we did not need to deal with when we proved curvature estimates for stable intrinsic balls in Chapter 2. We need cutoff functions for the "inner boundary" γ, the "outer boundary" $\{\rho = R_1\}$ and the "sides" (i.e., the geodesics perpendicular to $\partial\gamma$).

Define a function ψ on S_{R_1} by

$$\psi = \psi(\rho) = 1 - \rho/R_1\,.$$

Next, let γ_a and γ_b be the geodesics perpendicular to $\partial\gamma$ and set

$$d_S = \mathrm{dist}_{S_{R_1}}(\cdot, \gamma_a \cup \gamma_b)\,.$$

Define functions χ_1, χ_2 on S_{R_1} by

$$\chi_1 = \chi_1(d_S) = \begin{cases} d_S/r_1 & \text{if } 0 \leq d_S \leq r_1\,, \\ 1 & \text{otherwise}\,, \end{cases} \tag{8.27}$$

$$\chi_2 = \chi_2(\rho) = \begin{cases} \rho/r_1 & \text{if } 0 \leq \rho \leq r_1\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{8.28}$$

Set $\chi = \chi_1 \chi_2$. Since $B^{\Sigma}_{r_1/8}(x)$ is 1/2-stable for every point $x \in S_{R_1}$, the curvature estimate of Lemma 8.7 gives

$$\sup_{S_{R_1}} |A|^2 \leq C\, r_1^{-2} . \tag{8.29}$$

Using this and standard comparison theorems to bound the area of a tubular neighborhood of $\gamma \cup \gamma_a \cup \gamma_b$, we get

$$\mathrm{Area}(S_{R_1} \cap \{\chi < 1\}) \leq C' \, (R_1\, r_1 + m\, r_1^2) . \tag{8.30}$$

Combining this area bound with (8.29) gives

$$\int_{S_{R_1}} |\nabla \chi|^2 + \int_{S_{R_1} \cap \{\chi < 1\}} |A|^2 \leq \tilde{C} \left(\frac{R_1}{r_1} + m \right) . \tag{8.31}$$

At the end of the proof, we will need the following bound for χ_1 (which we used to get (8.31)):

$$\int_{S_{R_1}} |\nabla \chi_1|^2 + \int_{S_{R_1} \cap \{\chi_1 < 1\}} |A|^2 \leq \tilde{C} \, \frac{R_1}{r_1} . \tag{8.32}$$

Substituting $\chi\psi$ into the 1/2-stability inequality, the absorbing inequality $2ab \leq \epsilon\, a^2 + \frac{1}{\epsilon}\, b^2$ and (8.31) give

$$\begin{aligned} \frac{1}{2} \int |A|^2 \chi^2 \psi^2 &\leq \int |\nabla (\chi \psi)|^2 = \int \left(\chi^2 |\nabla \psi|^2 + 2\chi\, \psi \langle \nabla \chi, \nabla \psi \rangle + \psi^2 |\nabla \chi|^2 \right) \\ &\leq \tilde{C}\, (1 + 1/\epsilon) \left(\frac{R_1}{r_1} + m \right) + (1+\epsilon) \int \chi^2 |\nabla \psi|^2 \\ &\leq \tilde{C}\, (1 + 1/\epsilon) \left(\frac{R_1}{r_1} + m \right) + \frac{(1+\epsilon)}{R_1^2} \int_0^{R_1} \ell(t)\, dt , \end{aligned} \tag{8.33}$$

where all integrals, except the last, are over S_{R_1}.

Using that $(\psi^2)'' = 2/R_1^2$ and then integrating by parts (in one variable) twice gives

$$\begin{aligned} \frac{2}{R_1^2} \int_0^{R_1} \bar{K}(t) &= \int_0^{R_1} \bar{K}(t) (\psi^2)'' = - \int_0^{R_1} K(t) (\psi^2)' = \int_{S_{R_1}} |A|^2 \psi^2 \\ &\leq \tilde{C} \left(\frac{R_1}{r_1} + m \right) + \int_{S_{R_1}} |A|^2 \chi^2 \psi^2 \\ &\leq \tilde{C}\, (3 + 2/\epsilon) \left(\frac{R_1}{r_1} + m \right) + 2(1+\epsilon)\, R_1^{-2} \int_0^{R_1} \ell(t) , \end{aligned} \tag{8.34}$$

where the first inequality used (8.31) and the last inequality used (8.33).

(There is a slight abuse of notation in regarding ψ as a function on both $[0, R_1]$ and S_{R_1}. The third equality used the coarea formula and $|\nabla \rho| = 1$ to differentiate $K(t)$.)

Setting $\epsilon = \frac{1}{2}$ and combining (8.26) and (8.34) gives

$$\begin{aligned} \frac{4}{R_1^2} \int_0^{R_1} \ell(t) &\leq 12\,\pi\, m + 2\, C_0\, m + \frac{2}{R_1^2} \int_0^{R_1} \bar{K}(t) \\ &\leq (12\,\pi + 2\, C_0)\, m + \frac{3}{R_1^2} \int_0^{R_1} \ell(t) + 7\, \tilde{C} \left(\frac{R_1}{r_1} + m \right). \end{aligned} \tag{8.35}$$

The crucial point is that the integral of $\ell(t)$ turns up on both sides, but with a smaller constant on the right, so this gives a bound. Namely, (8.35) gives

$$R_1^{-2}\, \mathrm{Area}(S_{R_1}) = R_1^{-2} \int_0^{R_1} \ell(t) \leq C_4 \left(\frac{R_1}{r_1} + m(1 + C_0) \right). \tag{8.36}$$

Since $\ell(t)$ is monotone increasing (by (8.23)), (8.36) gives the second inequality in (8.19) for $t = 3\,R_1/4$. Since the above argument applies with R_1 replaced by t where $2\,r_1 < t < R_1$, we get (8.19) for $2\,r_1 \leq t \leq 3\,R_1/4$.

To prove the last claim, we will use the stability inequality together with the logarithmic cutoff trick to take advantage of the quadratic area growth. Define a cutoff function ψ_1 by

$$\psi_1 = \psi_1(\rho) = \begin{cases} \log(\rho/r_1)/\log \Omega & \text{on } S_{r_1, \Omega\, r_1}\,, \\ 1 & \text{on } S_{\Omega\, r_1, R_1/\Omega}\,, \\ -\log(\rho/R_1)/\log\Omega & \text{on } S_{R_1/\Omega, R_1}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{8.37}$$

Using (8.19) and (8.36) (as in (1.107)), we get

$$\int_{S_{R_1}} |\nabla \psi_1|^2 \leq \frac{C(C_0 + m + R_1/r_1)}{\log \Omega}. \tag{8.38}$$

As in (8.33), we apply the 1/2-stability inequality to $\chi_1 \psi_1$ to get

$$\begin{aligned} \frac{1}{2} \int |A|^2\, \chi_1^2\, \psi_1^2 &\leq \int_{S_{R_1}} |\nabla(\psi_1\, \chi_1)|^2 \leq 2 \int_{S_{R_1}} \left(|\nabla \psi_1|^2 + |\nabla \chi_1|^2 \right) \\ &\leq \frac{2\, C((C_0 + 1)\, m + R_1/r_1)}{\log \Omega} + \frac{2 \tilde{C}\, R_1}{r_1}. \end{aligned} \tag{8.39}$$

Finally, combining (8.32) and (8.39) gives (8.20). □

The next result from [**CM3**] shows that a large almost stable intrinsic sector must contain a multi-valued graph. The idea is that the previous lemma gives subsectors with small total curvature, thus locally graphical, and these contain multi-valued graphs if they are sufficiently "wide".

Proposition 8.8 (Colding-Minicozzi, [**CM3**]). *Given $\omega > 8, 1 > \epsilon > 0, C_0$, and N, there exist m_1, Ω_1 so that the following holds:*

If $0 \in \Sigma$ is an embedded minimal disk containing a curve $\gamma \subset \partial B^{\Sigma}_{r_1}$ with

- $\int_\gamma k_g < C_0\, m_1$ *and* $\mathrm{Length}(\gamma) = m_1\, r_1$,
- $\mathcal{T}_{r_1/8}(S_{\Omega_1^2\,\omega\, r_1}(\gamma))$ *is 1/2-stable and disjoint from* $\partial\Sigma$,

then (after a rotation of $\mathbb{R}^3$*)* $S_{\Omega_1^2\,\omega\, r_1}(\gamma)$ *contains an* N*-valued graph* Σ_N *over* $D_{\omega\,\Omega_1\, r_1} \setminus D_{\Omega_1\, r_1}$ *with gradient* $\leq \epsilon$, $|A| \leq \epsilon/r$, *and*

$$\mathrm{dist}_{S_{\Omega_1^2\,\omega\, r_1}(\gamma)}(\gamma, \Sigma_N) < 4\,\Omega_1\, r_1\,. \tag{8.40}$$

Proof. As in the previous proof, define a function ρ by

$$\rho(x) = \mathrm{dist}_{S_{\Omega_1^2\,\omega\, r_1}}(x, \gamma)\,.$$

We will choose $\Omega_1 > 12$ and then set $m_1 = \omega\,\Omega_1^2 \log \Omega_1$. By Lemma 8.7 (with $\Omega = \Omega_1/6$, $R_1 = \Omega_1^2\,\omega\, r_1$, and $m = 32\, m_1/3$),

$$\begin{aligned}\int_{S_{\frac{\Omega_1 r_1}{6}, 6\,\Omega_1\,\omega\, r_1}(\gamma)} |A|^2 &\leq C\left(\Omega_1^2\,\omega + \frac{m_1(1+C_0)}{\log(\Omega_1/6)}\right) \leq C\left(\Omega_1^2\,\omega + \frac{2\,m_1(1+C_0)}{\log \Omega_1}\right) \\ &= \frac{(3+2\,C_0)\,C\, m_1}{\log \Omega_1}\,. \end{aligned} \tag{8.41}$$

Partition γ into $m_2 \approx m_1/(8\pi\,(N+1))$ curves $\gamma_1, \dots, \gamma_{m_2} \subset \gamma$ with

$$\mathrm{Length}(\gamma_i) = 8\,\pi\,(N+1)\, r_1\,.$$

Since we have partitioned the sector over γ into m_2 subsectors whose interiors are pairwise disjoint, at least one of the γ_i's satisfies

$$\begin{aligned}\int_{S_{\frac{\Omega_1 r_1}{6}, 6\,\Omega_1\,\omega\, r_1}(\gamma_i)} |A|^2 &\leq \frac{1}{m_2} \int_{S_{\Omega_1\, r_1/6, 6\,\Omega_1\,\omega\, r_1}(\gamma)} |A|^2 \\ &\leq \frac{(3+2\,C_0)\,C\,8\,\pi\,(N+1)}{\log \Omega_1}\,,\end{aligned} \tag{8.42}$$

where the last inequality used (8.41).

Equation (8.42) gives a very flat sector as long as we choose Ω_1 much larger than N. We will use this flatness to get the desired N-valued graph.

Property (3) in Lemma 8.6 implies that

$$\int_{\gamma_i/2} k_g \geq \frac{1}{r_1}\,\mathrm{Length}(\gamma_1/2) = 4\,\pi\,(N+1)\,,$$

while property (5) in Lemma 8.6 implies that

$$\int_{S_{\Omega_1^2\,\omega\, r_1}(\gamma_i/2)\cap\{\rho=t\}} k_g$$

is monotone nondecreasing in t. Combining these, we can choose a connected curve $\tilde{\gamma} \subset \gamma_i/2$ with

$$(8.43) \qquad \int_{S_{\Omega_1^2 \omega r_1}(\tilde{\gamma}) \cap \{\rho = \Omega_1 r_1/3\}} k_g = 2\pi (N+1)\,.$$

Set $S = S_{\Omega_1 r_1/3, 3\Omega_1 \omega r_1}(\tilde{\gamma})$ and $\hat{\gamma} = S \cap \{\rho = \Omega_1 r_1/3\}$.

Next, we use the Gauss-Bonnet theorem, (8.42), and (8.43), (for Ω_1 large) to get that

$$(8.44) \quad 2\pi (N+1) \leq \int_{S \cap \{\rho = t\}} k_g \leq 2\pi (N+1) + \frac{1}{2} \int_S |A|^2 \leq 2\pi (N+2)\,.$$

Combining the upper bound for the total curvature with the pointwise lower bound $k_g \geq \frac{1}{\rho + r_1}$ from Lemma 8.6, gives the length bound

$$\text{Length}(S \cap \{\rho = t\}) \leq 2\pi (N+2)(t + r_1) \leq 4\pi (N+2)\, t\,.$$

Finally, observe that, by stability, (8.42), and using (2) in Lemma 8.6, the mean value theorem gives for $y \in S$,

$$(8.45) \qquad \sup_{\mathcal{B}_{\rho(y)/3}(y)} |A|^2 \leq \frac{C_1 N}{\rho^2(y) \log \Omega_1}\,.$$

Integrating (8.45) along rays and level sets of ρ, we get

$$(8.46) \qquad \max_{x,y \in S} \text{dist}_{\mathbf{S}^2}(N(x), N(y)) \leq \frac{C_2 (\log \omega + N)\sqrt{N}}{\sqrt{\log \Omega_1}}\,.$$

(The $\log \omega$ term comes from integrating over the rays, since the ratio of the outer to inner radii is $\approx \omega$. The other term is from integrating along a level set $\rho = t$ which has length at most $4\pi(N+2)\,t$ and $|A|$ at most $\approx \sqrt{N/\log \Omega_1}\, t^{-1}$.)

We can now combine these facts to get the proposition; this is quite similar to the proof of Theorem 2.18 in Chapter 2. Choose Ω_1 so that

$$\frac{C_2 (\log \omega + N)\sqrt{N}}{\sqrt{\log \Omega_1}} < C_3\, \epsilon\,,$$

where C_3 is a small constant to be chosen later. For C_3 sufficiently small, after rotating $\mathbb{R}^3$, S is locally a graph over $\{x_3 = 0\}$ with gradient $\leq \epsilon/2$. Since $\tilde{\gamma} \subset B_{2r_1}$ and $\Omega_1 > 12$, we have

$$\hat{\gamma} \subset B_{2r_1 + \Omega_1 r_1/3} \subset B_{\Omega_1 r_1/2}\,.$$

Choosing Ω_1 even larger and combining (3) in Lemma 8.6, (8.44), (8.45), and (8.46), we see that (the orthogonal projection) $\Pi(\hat{\gamma})$ is a convex planar curve with total curvature at least $(2N+1)\,\pi$, so that its Gauss map covers

$\mathbf{S}^1$ at least N times. Given $x \in \tilde{\gamma}$, set $\tilde{\gamma}_x = S \cap \gamma_x$. By (8.45), $\tilde{\gamma}_x$ has total (extrinsic geodesic) curvature at most

$$\frac{C_2 \sqrt{N} \, \log \omega}{\sqrt{\log \Omega_1}} < C_3 \, \epsilon$$

and hence $\tilde{\gamma}_x$ lies in a narrow cone centered on its tangent ray at $\tilde{x} = \tilde{\gamma}_x \cap \hat{\gamma}$. For C_3 small, this implies that $\tilde{\gamma}_x$ does not rotate and

$$|\Pi(\tilde{x}) - \Pi(\tilde{\gamma}_x \cap \{\rho = t\})| \geq \frac{9}{10} \left(t - \frac{\Omega_1 \, r_1}{3} \right) . \tag{8.47}$$

Hence, $\Pi(\partial \tilde{\gamma}_x \setminus \{\tilde{x}\}) \notin D_{2\,\omega\,\Omega_1\,r_1}$ which gives Σ_d and also

$$\mathrm{dist}_{S_{\Omega_1^2\,\omega\,r_1}(\gamma)}(\gamma, \Sigma_d) < 2\,\Omega_1\,r_1 \, .$$

□

1.4. The local structure near the axis. The key for proving Theorem 8.2 is to find n large intrinsic sectors with a scale-invariant curvature bound. To do this:

- We first use the curvature estimate for intrinsic balls in a minimal disk, i.e., Theorem 2.17, to bound $\mathrm{Length}(\partial B^{\Sigma}_R)/R$ from below for $R \geq R_0$.
- Corollary 8.4 gives $R_3 > R_0$ and n long disjoint curves $\tilde{\gamma}_i \subset \partial B^{\Sigma}_{R_3}$ so the sectors over $\tilde{\gamma}_i$ have bounded $\int |A|^2$.
- Theorem 2.17 gives the scale-invariant curvature bound.
- Once we have these sectors, for n large, two must be close in $\mathbb{R}^3$ and hence, by Lemmas 2.29 and 2.30 from Chapter 2, 1/2-stable.
- Finally, the N-valued graph is then given by corollary II.1.34 of [**CM3**] (see Proposition 8.8).

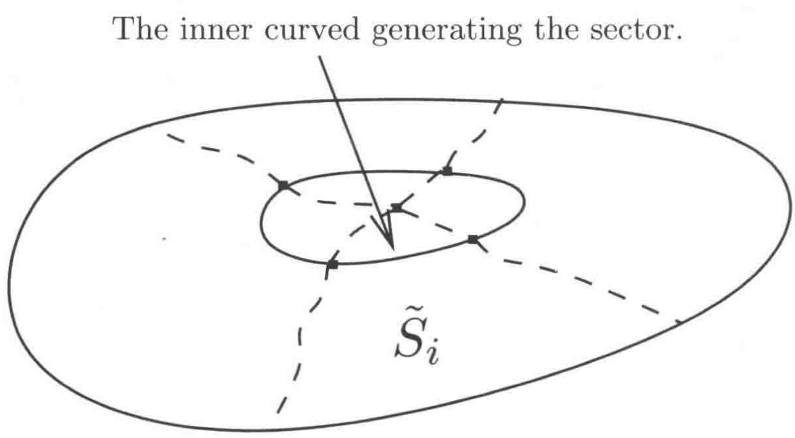

Figure 8.3. Equation (8.51) divides a punctured ball into sectors $\tilde{S}_i$.

Proof of Theorem 8.2. Rescale Σ by C/r_0 so that $|A|^2(0) = 1$ and $|A|^2 \leq 4$ on B_C.

Let C_2 be from Corollary 8.4 and then let $m_1, \Omega_1 > \pi$ be given by Proposition 8.8 with C_0 there $= 2\,C_2 + 2$. Fix a_0 large (to be chosen).

Since we have the lower bound $|A|(0) = 1$, the curvature estimate for embedded minimal disks in Theorem 2.17 implies that the total curvature in B_s^{Σ} can be made arbitrarily large by taking s sufficiently large. Using this and Corollary 2.7 to relate total curvature and the length, there exists $R_0 = R_0(a_0)$ so that for any $R_3 \geq R_0$,

$$a_0\, R_3 \leq R_3/4 \int_{B^{\Sigma}_{R_3/2}} |A|^2 \leq \text{Length}(\partial B^{\Sigma}_{R_3})\,. \tag{8.48}$$

Set $\beta = 2\Omega_1^2\,\omega$. Corollaries 2.7 and 8.4 give $R_2 = R_2(R_0, \beta)$ so if $C \geq R_2$, then there is $R_0 < R_3 < R_2/(2\,\beta)$ with

$$\begin{aligned}\int_{B^{\Sigma}_{3\,R_3}} |A|^2 + \beta^{-10} \int_{B^{\Sigma}_{2\,\beta\,R_3}} |A|^2 &\leq C_2\, {R_3}^{-2}\ \text{Area}(B^{\Sigma}_{R_3}) \\ &\leq C_2\, \frac{\text{Length}(\partial B^{\Sigma}_{R_3})}{2R_3}\,.\end{aligned} \tag{8.49}$$

Using (8.48), choose n so that

$$a_0\, R_3 \leq 4\, m_1\, n\, R_3 < \text{Length}(\partial B^{\Sigma}_{R_3}) \leq 8\, m_1\, n\, R_3\,, \tag{8.50}$$

and fix $2n$ disjoint curves $\tilde{\gamma}_i \subset \partial B^{\Sigma}_{R_3}$ with length $2\,m_1\,R_3$. Define the intrinsic sectors (see Figure 8.3)

$$\tilde{S}_i = \{\exp_0(v)\,|\,0 < |v| \leq 2\,\beta\, R_3 \text{ and } \exp_0(R_3\, v/|v|) \in \tilde{\gamma}_i\}\,. \tag{8.51}$$

Since the $\tilde{S}_i$'s are disjoint, combining (8.49) and (8.50) gives

$$\sum_{i=1}^{2n} \left(\int_{B^{\Sigma}_{3\,R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \right) \leq 4\, C_2\, m_1\, n\,. \tag{8.52}$$

Hence, after reordering the $\tilde{\gamma}_i$, we can assume that for $1 \leq i \leq n$,

$$\int_{B^{\Sigma}_{3\,R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \leq 4\, C_2\, m_1\,. \tag{8.53}$$

Using (2) in Lemma 8.6, there are curves $\gamma_i \subset \partial B^{\Sigma}_{2R_3} \cap \tilde{S}_i$ with length $2\,m_1\,R_3$ so that if $y \in S_i = S_{\beta R_3}(\gamma_i) \subset \tilde{S}_i$, then

$$B^{\Sigma}_{\text{dist}_{\Sigma}(0,y)/2}(y) \subset \tilde{S}_i\,. \tag{8.54}$$

Hence, by the curvature estimate, Theorem 2.17, for embedded disks and (8.53), we get for $y \in S_i$ and $i \leq n$,

$$\sup_{\mathcal{B}_{\text{dist}_{\Sigma}(0,y)/4}(y)} |A|^2 \leq C_3\, \text{dist}_{\Sigma}^{-2}(0,y)\,, \tag{8.55}$$

where $C_3 = C_3(\beta, m_1)$. For $i \leq n$, (8.53) and the Gauss-Bonnet theorem yield

$$\int_{\gamma_i} k_g \leq 2\,\pi + 2\,C_2\, m_1 < (2\,C_2 + 2)\, m_1\,. \tag{8.56}$$

By (8.55) and (4) in Lemma 8.6, there exists $C_4 = C_4(C_3)$ so that for $i \leq n$,

$$1/(2R_3) \leq \min_{\gamma_i} k_g \leq \max_{\gamma_i} k_g \leq C_4/R_3\,. \tag{8.57}$$

Applying Lemma 2.30 repeatedly (and using (8.55)), it is easy to see that there exists $\alpha > 0$ so that if $i_1 < i_2 \leq n$ and

$$\mathrm{dist}_{C^1([0,2m_1],\mathbb{R}^3)}\left(\frac{\gamma_{i_1}}{R_3}, \frac{\gamma_{i_2}}{R_3}\right) \leq \alpha\,, \tag{8.58}$$

then

$$\{z + u(z)\,N(z) \,|\, z \in \mathcal{T}_{R_3/4}(S_{i_1})\} \subset \cup_{y \in S_{i_2}} B^{\Sigma}_{\mathrm{dist}_\Sigma(0,y)/4}(y)$$

for a function $u \neq 0$ with

$$|\nabla u| + |A|\,|u| \leq C_0'\ \mathrm{dist}_{C^1([0,2m_1],\mathbb{R}^3)}\left(\frac{\gamma_{i_1}}{R_3}, \frac{\gamma_{i_2}}{R_3}\right)\,. \tag{8.59}$$

Here $\mathrm{dist}_{C^1([0,2m_1],\mathbb{R}^3)}\,(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3)$ is the scale-invariant C^1-distance between the curves.

Next, we use compactness to show that (8.58) must hold for n large. Namely, since each $\gamma_i/R_3 \subset B_2$ is parametrized by arclength on $[0, 2m_1]$ and has a uniform $C^{1,1}$ bound by (8.57), this set of maps is compact by the Arzela-Ascoli theorem. Hence, there exists n_0 so that if $n \geq n_0$, then (8.58) holds for some $i_1 < i_2 \leq n$. In particular, (8.59) and Lemma 2.29 imply that S_{i_1} is 1/2-stable for n large (now choose a_0, R_0, R_2). After rotating $\mathbb{R}^3$, Proposition 8.8 gives the N-valued graph $\Sigma_g \subset S_{i_1}$ over $D_{2\omega\,\Omega_1\,R_3} \setminus D_{2\Omega_1\,R_3}$ with gradient $\leq \epsilon$, $|A| \leq \epsilon/r$, and

$$\mathrm{dist}_\Sigma(0, \Sigma_g) \leq 8\,\Omega_1\,R_3\,.$$

Rescaling by r_0/C, the theorem follows with $\bar{R} = 2\Omega_1\,R_3 r_0/C$. □

2. One-sided Curvature Estimate

One of the key tools used to understand embedded minimal surfaces is the one-sided curvature estimate of [**CM6**]. The two sections that follow will use this estimate to prove two very different long-standing conjectures, the Generalized Nitsche Conjecture and the Calabi-Yau Conjectures for embedded surfaces.

The one-sided curvature estimate roughly states that an embedded minimal disk that lies on one side of a plane, but comes close to the plane, has bounded curvature. Alternatively, it says that if the curvature is large at

the center of a ball, then the minimal disk propagates out in all directions so that it cannot be contained on one side of any plane that passes near the center of the ball.

Theorem 8.9 (Colding-Minicozzi, [**CM6**]). *There exists $\epsilon_0 > 0$ so that the following holds. Let $y \in \mathbb{R}^3$, $r_0 > 0$ and*

$$\Sigma^2 \subset B_{2r_0}(y) \cap \{x_3 > x_3(y)\} \subset \mathbb{R}^3 \tag{8.60}$$

be a compact embedded minimal disk with $\partial\Sigma \subset \partial B_{2\,r_0}(y)$. For any connected component Σ' of $B_{r_0}(y) \cap \Sigma$ with $B_{\epsilon_0\,r_0}(y) \cap \Sigma' \neq \emptyset$,

$$\sup_{\Sigma'} |A_{\Sigma'}|^2 \leq r_0^{-2}\,. \tag{8.61}$$

The one-sided curvature estimate is proven in [**CM6**], using results from [**CM3**]-[**CM6**].

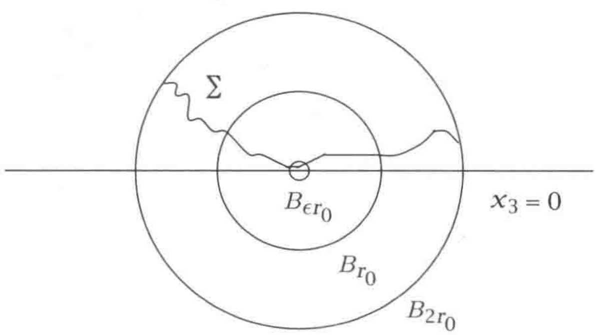

Figure 8.4. The one-sided curvature estimate.

The example of a rescaled catenoid shows that simply connected and embedded are both essential hypotheses for the one-sided curvature estimate. More precisely, the height of the catenoid grows logarithmically in the distance to the axis of rotation. In particular, the intersection of the catenoid with B_{r_0} lies in a slab of thickness $\approx \log r_0$ and the ratio of

$$\frac{\log r_0}{r_0} \to 0 \text{ as } r_0 \to \infty\,. \tag{8.62}$$

Thus, after dilating the catenoid by $\frac{1}{j}$, we get a sequence of minimal surfaces in the unit ball that converges as sets to $\{x_3 = 0\}$ as $j \to \infty$. However, the catenoid is not flat, so these rescaled catenoids have $|A| \to \infty$ and (8.61) does not apply for j large.

We will later come back to an intrinsic version of the one-sided curvature estimate that was proven in [**CM24**].

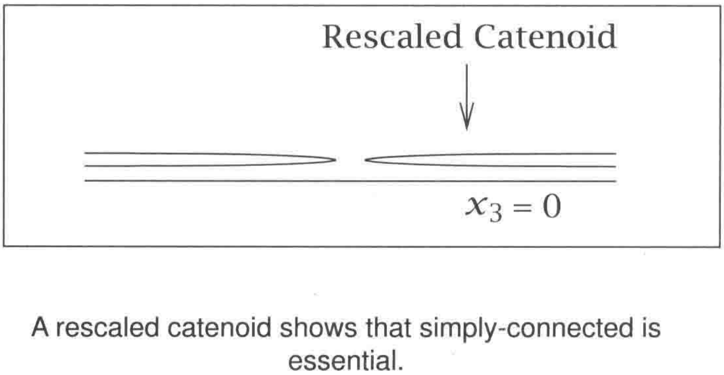

Figure 8.5. Rescaled catenoids.

3. Generalized Nitsche Conjecture

This section studies the ends of a properly embedded complete minimal surface $\Sigma^2 \subset \mathbb{R}^3$ with finite topology. In particular, we prove the Generalized Nitsche Conjecture as an application (from [**CM10**]) of the one-sided curvature estimate. The original proof is due to Collin, [**Co**].

Given a properly embedded minimal surfaces with finite topology, each end has a representative E which is a properly embedded minimal annulus. If E has finite total curvature, it is asymptotic to either a plane or half of a catenoid. On the other hand, the helicoid provides the only known example of an end with finite topology and infinite total curvature. Clearly, no representative for the end of the helicoid can be disjoint from an end of a plane or catenoid.

The main result of [**CM10**] is that any complete properly embedded minimal annulus which lies above a sufficiently narrow downward sloping cone must have finite total curvature. This is closely related to a result of P. Collin [**Co**] described below.

In [**HoMe2**], D. Hoffman and W. Meeks proved that Σ has at most two ends with infinite total curvature. Further, they conjectured that if Σ as above has at least two ends, then Σ must have finite total curvature (this is the "finite total curvàture conjecture"; cf. [**Me1**]).

If Σ has at least two ends, then there is either an end of a plane or a catenoid disjoint from Σ by lemma 5 of [**HoMe2**]. Therefore, to prove the finite total curvature conjecture, it suffices to show that a properly embedded minimal annular end E which lies above the bottom half of a catenoid has finite total curvature.

In this direction, W. Meeks and H. Rosenberg [**MeR1**] showed that if Σ has at least two ends, then Σ is *conformally* equivalent to a compact

Riemann surface with finitely many points removed. Using this, they showed that an annular end E arising in the finite total curvature conjecture lies in a half-space. In fact, they showed that E is either asymptotically planar (with finite total curvature) or satisfies the hypotheses of the "generalized Nitsche conjecture."

Conjecture 8.10 (Generalized Nitsche Conjecture, [**MeR1**]; Collin's Theorem, [**Co**]). *For $t \geq 0$ let $P_t = \{x_3 = t\} \subset \mathbb{R}^3$. Suppose that $E \subset \{x_3 \geq 0\}$ is a properly embedded minimal annulus with $\partial E \subset P_0$ and that E intersects every P_t for $t > 0$ in a simple closed curve; then E has finite total curvature.*

P. Collin proved this conjecture in [**Co**], thereby showing, using [**MeR1**], that, for properly embedded complete minimal surfaces with at least two ends, finite topology is equivalent to finite total curvature. An example of a properly immersed minimal cylinder in $\mathbb{R}^3$ with infinite total curvature, constructed by H. Rosenberg and E. Toubiana [**RoTo**], shows that embeddedness is a necessary hypothesis in both the finite total curvature and generalized Nitsche conjectures. Well-known examples show that properness is also necessary.

Let x_1, x_2, x_3 be the standard coordinates on $\mathbb{R}^3$. Given any $\epsilon \in \mathbb{R}$ we let $\mathcal{C}_\epsilon \subset \mathbb{R}^3$ denote the conical region

$$\mathcal{C}_\epsilon = \left\{ x_3 > \epsilon \sqrt{x_1^2 + x_2^2} \right\} . \tag{8.63}$$

With this definition, $\mathcal{C}_0$ is a half-space and $\mathcal{C}_\epsilon$ is convex if and only if $\epsilon \geq 0$.

Theorem 8.11 (Colding-Minicozzi [**CM10**]). *There exists $\epsilon > 0$ such that any complete properly embedded minimal annular end $E \subset \mathcal{C}_{-\epsilon}$ has finite total curvature.*

The generalized Nitsche conjecture follows directly from Theorem 8.11 since $\{x_3 \geq 0\} = \mathcal{C}_0 \subset \mathcal{C}_{-\epsilon}$. In fact, since the height of the catenoid only grows logarithmically, the finite total curvature conjecture also follows directly from Theorem 8.11.

The one-sided curvature estimate, Theorem 8.9, will be the key to the proof.

The following lemma will be used in the proof to find "low points" in the end E.

Lemma 8.12. *With E as in the theorem above and any $\delta > 0$, there exists a sequence of points $y_j \in E \setminus \mathcal{C}_\delta$ with $|y_j| \to \infty$.*

Proof. If this fails for $\delta > 0$, we may rescale to get $\partial E \subset B_1$ and

$$E \setminus B_1(0) \subset \mathcal{C}_\delta . \tag{8.64}$$

Since E is proper and $\delta > 0$, a curve $\sigma_0 \subset E$ connects B_1 to $\{x_3 = 200\}$.

Since $\delta > 0$, there exists $s_0 < 0$ so that $\cosh^{-1}(t) + s_0 < \delta t$ for all $t \geq 1$; hence

$$\mathrm{Cat}(0, 0, s_0) \cap \bar{E} = \emptyset \, . \tag{8.65}$$

By the strong maximum principle, there cannot be a first $s > s_0$ with $\mathrm{Cat}(0, 0, s) \cap \bar{E} \neq \emptyset$ (since $\partial E \subset B_1$). In particular,

$$E \subset \{x_1^2 + x_2^2 < 1\} \, . \tag{8.66}$$

Again by the strong maximum principle, there cannot be a first $0 < s \leq 10$ so that

$$\mathrm{Cat}(s, 0, 100) \cap \bar{E} \neq \emptyset \, . \tag{8.67}$$

This gives the contradiction since $\mathrm{Cat}(10, 0, 100)$ separates B_1 from $\{x_3 = 200\}$ in $\{x_1^2 + x_2^2 < 1\}$. □

In the proof of the theorem, we will use intrinsic balls in addition to the usual extrinsic balls $B_s(y)$. The intrinsic ball in E of radius s about a point $y \in E$ will be denoted $\mathcal{B}_s(y)$.

Proof of Theorem 8.11. By elementary topology, there is a connected curve $\beta \subset \mathcal{C}_{-\epsilon} \setminus E$ with one endpoint in ∂E and the other at the origin. Since the statement of the theorem is scale invariant, we may suppose that

$$\partial E \cup \beta \subset B_1 \cap \mathcal{C}_{-\epsilon} \, . \tag{8.68}$$

Lemma 8.12 yields a sequence of points $y_j \in E \setminus \mathcal{C}_\epsilon$ with $4 < |y_j| \to \infty$. By Sard's theorem, we may choose a sequence r_j with

$$|y_j| - 2 \leq 6 \, r_j \leq |y_j| - 1 \tag{8.69}$$

so that $B_{r_j}(y_j)$ and $B_{2 r_j}(y_j)$ intersect E transversely.

For each j, we define a function f_j on $\mathbb{R}^3$ by

$$f_j(x) = 2 \, (x \cdot y_j - 2 \, |y_j|)^2 - | \, |y_j| \, x - 2 \, y_j|^2 \, , \tag{8.70}$$

and note that f_j is superharmonic on any minimal surface, so that

$$\Delta_E \, f_j \leq 0 \, . \tag{8.71}$$

Let Σ_j denote the connected component of $B_{2 r_j}(y_j) \cap E$ containing y_j. If Σ_j is not a disk, then there exists a minimal annulus Γ_j with boundary components $\eta_j^1 \subset B_1$ and $\eta_j^2 \subset B_{2 r_j}(y_j)$. Since f_j is positive on η_j^1 and η_j^2, the maximum principle implies $f_j > 0$ on Γ_j. This is a contradiction since η_j^1 and η_j^2 are in different connected components of $\{f_j > 0\}$, and hence Σ_j is a disk.

By construction, we have $x_3(y_j) \leq 12 \, \epsilon \, r_j$ and

$$B_{2 r_j}(y_j) \cap \Sigma_j \subset B_{2 r_j}(y_j) \cap \mathcal{C}_{-\epsilon} \subset \{x_3 \geq -14 \, \epsilon \, r_j\} \, . \tag{8.72}$$

We conclude that the one-sided curvature estimate, Theorem 8.9, applies. In particular, if $26\,\epsilon \leq \epsilon_0$, then

$$\sup_{\Sigma_j^1} |A_{\Sigma_j^1}|^2 \leq r_j^{-2}\,, \tag{8.73}$$

where Σ_j^1 is the connected component of $B_{r_j}(y_j) \cap \Sigma_j$ containing y_j.

The curvature estimate (8.73) allows us to apply the Harnack inequality, [**CgYa**], to the positive harmonic function $x_3 + 14\,\epsilon\, r_j$ on $\mathcal{B}_{r_j}(y_j) \subset \Sigma_j^1$. We get that

$$\sup_{\mathcal{B}_{\frac{3r_j}{4}}(y_j)} (x_3 + 14\,\epsilon\, r_j) \leq C_h\,(x_3 + 14\,\epsilon\, r_j)(y_j) \leq 26\, C_h\,\epsilon\, r_j\,, \tag{8.74}$$

where C_h comes from the Harnack estimate. Applying the gradient estimate, we have

$$\sup_{\mathcal{B}_{\frac{5\,r_j}{8}}(y_j)} |\nabla x_3| \leq C_g\,(26\, C_h\,\epsilon)\,, \tag{8.75}$$

where C_g comes from the gradient estimate, [**CgYa**]. For $\epsilon > 0$ small, (8.75) implies that $\mathcal{B}_{\frac{5\,r_j}{8}}(y_j)$ is a graph with small gradient over $x_3 = 0$. In particular, there is a point

$$y_j^1 \in \partial B_{|y_j|} \cap \{(x_1 - x_1(y_j))^2 + (x_2 - x_2(y_j))^2 = r_j^2/4\} \cap \mathcal{B}_{\frac{5\,r_j}{8}}(y_j)\,. \tag{8.76}$$

By (8.74), if $26\, C_h\,\epsilon \leq \epsilon_0$, we may now apply the preceding argument with y_j^1 in place of y_j.

After iterating this at most $48\,\pi + 1$ times, we go entirely around the cylinder

$$\partial B_{|y_j|} \cap \left\{-\epsilon\,|y_j| \leq x_3 \leq C_h^{48\,\pi+1}\,(6\,\epsilon\,|y_j|)\right\}\,. \tag{8.77}$$

If $\epsilon > 0$ is small enough, then the one-sided curvature estimate applies in this entire cylinder. Therefore, so long as ϵ is sufficiently small, iterating this gives a curve

$$\gamma_j \subset \partial B_{|y_j|} \cap \left\{-\epsilon\,|y_j| \leq x_3 \leq C_h^{48\,\pi+1}\,(6\,\epsilon\,|y_j|)\right\} \cap E\,, \tag{8.78}$$

so that γ_j is graphical over $\{x_3 = 0\}$. Since γ_j is embedded, it either spirals indefinitely or is closed and linked with the x_3-axis. Since E is proper, γ_j is compact and hence must be closed.

Let $E_j \subset E$ be the connected component of $B_{|y_j|} \cap E$ containing ∂E. By the maximum principle, E_j is an annulus and the other components of $B_{|y_j|} \cap E$ are disks.

We will show next that $\partial E_j = \partial E \cup \sigma_j$ where

$$\sigma_j \subset \partial B_{|y_j|} \cap \left\{-\epsilon\,|y_j| \leq x_3 \leq C_h^{48\,\pi+1}\,(6\,\epsilon\,|y_j|)\right\}\,. \tag{8.79}$$

This is immediate if $\gamma_j \subset \overline{E_j}$ by (8.78). On the other hand, if $\gamma_j \not\subset \overline{E_j}$, then the component of $B_{|y_j|} \cap E$ bounded by γ_j must be a disk $F_j \subset E$. In this case, (8.78) and the maximum principle imply that

$$F_j \subset \left\{-\epsilon\,|y_j| \leq x_3 \leq C_h^{48\,\pi+1}\,(6\,\epsilon\,|y_j|)\right\} . \tag{8.80}$$

Since ∂F_j is graphical and F_j is a disk, (8.80) implies that F_j separates $\partial \mathcal{C}_{-\epsilon}$ and $\{x_3 > C_h^{48\,\pi+1}\,(6\,\epsilon\,|y_j|)\}$ in $B_{|y_j|}$. Since E is embedded and $F_j \cap \beta = \emptyset$, we conclude that (8.79) holds.

As above, (8.79) allows us to apply Theorem 8.9 to prove that σ_j is graphical with uniformly small gradient and, furthermore, that there is a uniform scale-invariant bound on $|A|$ on a neighborhood of the graph σ_j. As in the proof of the positive mass theorem in the previous chapter, this implies a uniform bound for the total geodesic curvature of σ_j. Thus, since the other boundary component ∂E of E_j is independent of j and every E_j has the same topology, the Gauss-Bonnet theorem (with boundary) gives that

$$\int_{E_j} |A|^2 \leq C_1 \,, \tag{8.81}$$

where $C_1 < \infty$ and E_j^0 does not depend on j. Since the E_j's exhaust E as $j \to \infty$, (8.81) and the monotone convergence theorem imply that

$$\int_{E} |A|^2 \leq C_1 \,, \tag{8.82}$$

completing the proof. □

4. Calabi-Yau Conjectures for Embedded Surfaces

Recall that an immersed submanifold in $\mathbb{R}^n$ is proper if the preimage of any compact subset of $\mathbb{R}^n$ is compact in the surface. This property has played an important role in the theory of minimal submanifolds and many of the classical theorems in the subject assume properness. It is easy to see that any compact submanifold is automatically proper. On the other hand, there is no reason to expect a general immersion (or even embedding) to be proper. For example, the noncompact curve parametrized in polar coordinates by

$$\rho(t) = \pi + \arctan(t) \,,$$
$$\theta(t) = t$$

spirals infinitely between the circles of radius $\pi/2$ and $3\pi/2$. However, it was long thought that a minimal immersion (or embedding) should be better behaved. This principle was captured by the Calabi-Yau conjectures, dating back to the 1960s. Much work has been done on them over the past four decades. Their original form was given in 1965 in [**Ce**] where E. Calabi made

the following two conjectures about minimal surfaces (see also S.S. Chern, page 212 of [**Ch1**] and S.T. Yau's 1982 problem list, [**Ya4**]):

Conjecture 8.13. *"Prove that a complete minimal hypersurface in $\mathbb{R}^n$ must be unbounded."*

Calabi continued: "It is known that there are no compact minimal submanifolds of $\mathbb{R}^n$ (or of any simply connected complete Riemannian manifold with sectional curvature ≤ 0). A more ambitious conjecture is":

Conjecture 8.14. *"A complete [nonflat] minimal hypersurface in $\mathbb{R}^n$ has an unbounded projection in every $(n-2)$-dimensional flat subspace."*

The immersed versions of these conjectures turned out to be false. Namely, Jorge and Xavier, [**JXa2**], constructed nonflat minimal immersions contained between two parallel hyperplanes in 1980, giving a counterexample to the immersed version of the more ambitious Conjecture 8.14; see also [**RoTo**]. Another significant development came in 1996, when N. Nadirashvili, [**Na1**], constructed a complete immersion of a minimal disk into the unit ball in $\mathbb{R}^3$, showing that Conjecture 8.13 also failed for immersed surfaces; see [**MaMo1**], [**LMaMo1**], [**LMaMo2**], for other topological types than disks.

It is clear from the definition of proper that a proper minimal surface in $\mathbb{R}^3$ must be unbounded, so the examples of Nadirashvili are not proper. Much less obvious is that the plane is the only complete proper immersed minimal surface in a halfspace. This is, however, a consequence of the strong half-space theorem of D. Hoffman and W. Meeks:

Theorem 8.15 (Hoffman-Meeks, [**HoMe3**]). *A complete connected properly immersed minimal surface contained in $\{x_3 > 0\} \subset \mathbb{R}^3$ must be a horizontal plane $\{x_3 = \text{Constant}\}$.*

In [**CM24**], it was shown that the Calabi-Yau Conjectures were true for embedded surfaces. We will describe this more precisely below.

The main result of [**CM24**] is an effective version of properness for disks, giving a chord-arc bound. Obviously, intrinsic distances are larger than extrinsic distances, so the significance of a chord-arc bound is the reverse inequality, i.e., a bound on intrinsic distances from above by extrinsic distances. This is accomplished in the next theorem:

Theorem 8.16 (Colding-Minicozzi, [**CM24**]). *There exists a constant $C > 0$ so that if $\Sigma \subset \mathbb{R}^3$ is an embedded minimal disk, $B^{\Sigma}_{2R} = B^{\Sigma}_{2R}(0)$ is an intrinsic ball in $\Sigma \setminus \partial \Sigma$ of radius $2R$, and if $\sup_{B^{\Sigma}_{r_0}} |A|^2 > r_0^{-2}$ where $R > r_0$, then for $x \in B^{\Sigma}_R$,*

$$C \operatorname{dist}_{\Sigma}(x, 0) < |x| + r_0 \,. \tag{8.83}$$

The assumption of a lower curvature bound, $\sup_{B^{\Sigma}_{r_0}} |A|^2 > r_0^{-2}$, in the theorem is a necessary normalization for a chord-arc bound. This can easily be seen by rescaling and translating the helicoid.

Properness of a complete embedded minimal disk is an immediate consequence of Theorem 8.16. Namely, by (8.83), as intrinsic distances go to infinity, so do extrinsic distances. Precisely, if Σ is flat, and hence a plane, then obviously Σ is proper and if it is nonflat, then $\sup_{B^{\Sigma}_{r_0}} |A|^2 > r_0^{-2}$ for some $r_0 > 0$ and hence Σ is proper by (8.83).

A consequence of Theorem 8.16 together with the one-sided curvature estimate of [**CM6**] (i.e., theorem 0.2 in [**CM6**]) is the following version of that estimate for intrinsic balls:

Corollary 8.17 (Colding-Minicozzi, [**CM24**]). *There exists $\epsilon > 0$, so that if*

$$\Sigma \subset \{x_3 > 0\} \subset \mathbb{R}^3 \tag{8.84}$$

is an embedded minimal disk, $B^{\Sigma}_{2R}(x) \subset \Sigma \setminus \partial\Sigma$, and $|x| < \epsilon\, R$, then

$$\sup_{B^{\Sigma}_{R}(x)} |A_{\Sigma}|^2 \leq R^{-2}\,. \tag{8.85}$$

As a corollary of this intrinsic one-sided curvature estimate we get that the second, and "more ambitious", of Calabi's conjectures is also true for embedded minimal disks.

In fact, [**CM24**] proved both of Calabi's conjectures and properness also for embedded surfaces with finite topology. Recall that a surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface with a finite set of points removed or "punctures". Each puncture corresponds to an end of Σ.

The following generalization of the half-space theorem gives Calabi's second, more ambitious, conjecture for embedded surfaces with finite topology:

Theorem 8.18 (Colding-Minicozzi, [**CM24**]). *The plane is the only complete embedded minimal surface with finite topology in $\mathbb{R}^3$ in a half-space.*

Theorem 8.19 (Colding-Minicozzi, [**CM24**]). *A complete embedded minimal surface with finite topology in $\mathbb{R}^3$ must be proper.*

There have been several properness results for Riemannian three-manifolds. W. Meeks and H. Rosenberg, [**MeR3**], generalized this to get a local version in Riemannian three-manifolds; they also extended it to embedded minimal surfaces with finite genus and positive injectivity radius in $\mathbb{R}^3$. In [**Cb**], B. Coskunuzer proved properness for area minimizing disks in hyperbolic three-space, assuming that there is at least one C^1 point in the boundary at infinity.

There has been extensive work on both properness and the half-space property assuming various curvature bounds. Jorge and Xavier, [**JXa1**] and [**JXa2**], showed that there cannot exist a complete immersed minimal surface with bounded curvature in $\bigcap_i \{x_i > 0\}$; later Xavier proved that the plane is the only such surface in a half-space, [**Xa**]. Recently, G.P. Bessa, Jorge and G. Oliveira-Filho, [**BJO**], and H. Rosenberg, [**Ro**], have shown that if a complete embedded minimal surface has bounded curvature, then it must be proper. This properness was extended to embedded minimal surfaces with locally bounded curvature and finite topology by Meeks and Rosenberg in [**MeR2**]; finite topology was subsequently replaced by finite genus in [**MePRs1**] by Meeks, J. Perez and A. Ros.

Inspired by Nadirashvili's examples, F. Martin and S. Morales constructed in [**MaMo2**] a complete bounded minimal immersion which is proper in the (open) unit ball. That is, the preimages of compact subsets of the (open) unit ball are compact in the surface and the image of the surface accumulates on the boundary of the unit ball. They extended this in [**MaMo3**] to show that any convex, possibly noncompact or nonsmooth, region of $\mathbb{R}^3$ admits a proper complete minimal immersion of the unit disk. There are a number of interesting related results, including [**AFM**], [**MaMeNa**], and [**ANa**].

4.1. Higher dimensions. In higher dimensions, very little is known about the Calabi-Yau Conjectures for hypersurfaces. The counterexamples to the immersed versions all rely upon Weierstrass techniques that do not extend to higher dimensions and there are no known counterexamples to Calabi's first conjecture. However, we conjecture that such counterexamples exist:

Conjecture 8.20. *For all $n > 3$, there is a complete immersed minimal hypersurface contained in the unit ball of $\mathbb{R}^n$.*

For embedded hypersurfaces, at least assuming some bounds on topology as in [**CM24**], we conjecture the following:

Conjecture 8.21. *If $n > 3$ and $\Sigma \subset \mathbb{R}^n$ is a complete embedded minimal hypersurface that is contractible, then it is unbounded.*

Moreover, we conjecture that properness holds:

Conjecture 8.22. *If $n > 3$ and $\Sigma \subset \mathbb{R}^n$ is a complete embedded minimal hypersurface that is contractible, then it is properly embedded.*

These conjectures are already very interesting when $n = 4$.

Higher dimensional catenoids show that Calabi's more ambitious conjecture does not hold when $n > 3$. Higher dimensional catenoids are embedded, are not flat, but are contained between two parallel planes. The difference

between $n = 3$ and $n > 3$ is essentially coming from the difference in harmonic function theory between surfaces and higher dimensions. Namely, the Green's function goes to infinity at infinity on $\mathbb{R}^2$, but it remains bounded on $\mathbb{R}^k$ for $k > 2$. This is relevant because the coordinate functions are harmonic.

4.2. Outline of the proof of properness. The proof of the Calabi-Yau conjectures for embedded surfaces is beyond the scope of this book, but we will give an idea of the proof using the one-sided curvature estimate in addition to some of the results on almost-stable minimal surfaces that we have described earlier.

The proof that complete embedded minimal disks are proper in [**CM24**] consists roughly of the following three main steps:

(1) Show that if the surface is compact in a ball, then in this ball we have good chord-arc bounds.
(2) Show that if each component of the intersection of each ball of a certain size is compact (so that by the first step we have good estimates), then each intersection with double the Euclidean balls is also compact. Initially possibly with a much worse constant but then by the first step with a good constant.
(3) Iterate the above two steps.

Step 1 above relies on [**CM3**]–[**CM6**] for properly embedded minimal disks. We will outline here the proof of step 2 assuming step 1.

Suppose, therefore, that all intersections of the given disk with all Euclidean balls of radius r are compact and have good chord-arc bounds. We will show the same for all Euclidean balls of radius $2r$.

If not; then there are two points x, $y \in B_{2r} \cap \Sigma$ in the same connected component of $B_{2r} \cap \Sigma$ but with $\text{dist}_\Sigma(x, y) \geq C\, r$ for some large constant C. Let γ be an intrinsic geodesic in $B_{2r} \cap \Sigma$ connecting x and y. By dividing γ into segments, we conclude that there must be a pair of points x_0 and y_0 on γ in B_{2r} which are intrinsically far apart yet extrinsically close. We will start at these two points and build out showing that x_0 and y_0 could not connect in $B_{2r} \cap \Sigma$. This will be the desired contradiction.

By the assumption, each component of $B_r(x_0) \cap \Sigma$ is compact and by step 1 has good chord-arc bounds; hence x_0 and y_0 must lie in different components. Thus we have two compact components of $B_r(x_0) \cap \Sigma$ which are extrinsically close near the center. Earlier results (the one-sided curvature estimate of [**CM6**]; see theorem 0.2 there) show that half of each of these two components must have curvature bounds. Since this bound for the

curvature is in terms of the size of the relevant balls, then it follows that a fixed fraction of these components must be almost flat — again relative to its size. In fact, it now follows easily that these two almost flat regions contain intrinsic balls centered at x_0 and y_0 and with radii a fixed fraction of r. We can, therefore, go to the boundary of these almost flat intrinsic balls and find two points x_1 and y_1; one point in each intrinsic ball which are extrinsically close yet intrinsically far apart.

Repeat the argument with x_1 and y_1 in place of x_0 and y_0 to get points x_2 and y_2. Iterating gives large regions in the surface centered at x_0 and y_0 with a priori curvature bounds. Once we have a priori curvature bounds then improvements involving stability show that even these large regions are almost flat and thus could not combine in B_{2r}. This is the desired contradiction and hence completes the outline of step 2 above of the proof that embedded minimal disks are proper.

4.3. Chord-arc assuming bounded curvature. In the overview of step 2 of the proof of the Calabi-Yau Conjectures, the one-sided curvature estimate came in naturally to give a curvature bound for the nearby, but disjoint, pieces of Σ. The next lemma shows how to use this curvature bound to prove the desired "chord-arc" type result (relating extrinsic and intrinsic distances).

Given a surface Σ containing 0 and a radius $R > 0$, we let $\Sigma_{0,R}$ denote the connected component of $B_R(0) \cap \Sigma$ that contains 0.

Lemma 8.23 (Colding-Minicozzi, lemma II.2.1 in [**CM6**]). *Given R_0, there exists R_1 so that the following holds:*

If $0 \in \Sigma \subset B_{R_1}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{R_1}$ and

$$\sup_{B^{\Sigma}_{R_1}} |A|^2 \leq 4\,, \tag{8.86}$$

then

$$\Sigma_{0,R_0} \subset B^{\Sigma}_{R_1}\,. \tag{8.87}$$

Proof. Let $\tilde{\Sigma}$ be the universal cover of Σ and

$$\tilde{\Pi} : \tilde{\Sigma} \to \Sigma \tag{8.88}$$

the covering map. With the definition of δ-stable as in section 2 of [**CM4**] (see Definition 2.27), the argument of [**CM2**] (i.e., curvature estimates for 1/2-stable surfaces; cf. Theorem 2.10) gives $C > 10$ so that if $B^{\Sigma}_{CR_0/2}(\tilde{z}) \subset \tilde{\Sigma}$ is 1/2-stable and $\tilde{\Pi}(\tilde{z}) = z$, then

$$\tilde{\Pi} : B^{\Sigma}_{5R_0}(\tilde{z}) \to B^{\Sigma}_{5R_0}(z) \tag{8.89}$$

is one-to-one and $B^{\Sigma}_{5R_0}(z)$ is a graph with

$$B_{4R_0}(z) \cap \partial B^{\Sigma}_{5R_0}(z) = \emptyset \,. \tag{8.90}$$

Corollary 2.13 in [**CM4**] (see Corollary 2.31) gives $\epsilon = \epsilon(CR_0) > 0$ so that if $|z_1 - z_2| < \epsilon$ and $|A|^2 \leq 4$ on (the disjoint balls) $B^{\Sigma}_{CR_0}(z_i)$, then each

$$B^{\Sigma}_{CR_0/2}(\tilde{z}_i) \subset \tilde{\Sigma} \tag{8.91}$$

is 1/2-stable where $\tilde{\Pi}(\tilde{z}_i) = z_i$.

We claim that there exists n so that

$$\Sigma_{0,R_0} \subset B^{\Sigma}_{(2n+1)\,CR_0} \,. \tag{8.92}$$

Suppose not; we get a curve $\sigma \subset \Sigma_{0,R_0} \subset B_{R_0}$ from 0 to $\partial B^{\Sigma}_{(2n+1)\,CR_0}$. For $i = 1, \dots, n$, fix points $z_i \in \partial B^{\Sigma}_{2i\,CR_0} \cap \sigma$. It follows that the intrinsic balls $B^{\Sigma}_{CR_0}(z_i)$:

- are disjoint;
- have centers in $B_{R_0} \subset \mathbb{R}^3$;
- have $|A|^2 \leq 4$.

In particular, there exist i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C'\, R_0\, n^{-1/3} < \epsilon \,, \tag{8.93}$$

and, by corollary 2.13 in [**CM4**] (see Corollary 2.31), each $B^{\Sigma}_{CR_0/2}(\tilde{z}_{i_j}) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_{i_j}) = z_{i_j}$. By (8.90), each $B^{\Sigma}_{5R_0}(z_{i_j})$ is a graph with

$$B_{4R_0}(z_{i_j}) \cap \partial B^{\Sigma}_{5R_0}(z_{i_j}) = \emptyset \,. \tag{8.94}$$

In particular,

$$B_{R_0} \cap \partial B^{\Sigma}_{5R_0}(z_{i_j}) = \emptyset \,. \tag{8.95}$$

This contradicts that $\sigma \subset B_{R_0}$ connects z_{i_j} to $\partial B^{\Sigma}_{CR_0}(z_{i_j})$. □

5. Embedded Minimal Surfaces with Finite Genus

We next describe two main structure theorems from [**CM7**] for "nonsimply connected" embedded minimal surfaces with finite genus. These results, as well as the results for disks described earlier, have played a key role in much of the recent development in minimal surface theory. We will first discuss planar domains (i.e., genus zero) and then turn to the case of finite genus.

5.1. Uniformly locally simply connected (ULSC). Sequences of planar domains that are not simply connected are, after passing to a subsequence, naturally divided into two separate cases depending on whether or not the topology is concentrating at points. To distinguish between these cases, we will say that a sequence of surfaces $\Sigma_i^2 \subset \mathbb{R}^3$ is *uniformly locally simply connected* (or ULSC) if for each $x \in \mathbb{R}^3$, there exists a constant $r_0 > 0$ (depending on x) so that for all $r \leq r_0$, and every surface Σ_i,

(8.96) each connected component of $B_r(x) \cap \Sigma_i$ is a disk.

For instance, a sequence of rescaled helicoids is ULSC, whereas a sequence of rescaled catenoids where the necks shrink to zero is not. Applying the above structure theory of [**CM3**]–[**CM6**] for disks to ULSC sequences gives that there are only two local models for such surfaces. That is, locally in a ball in $\mathbb{R}^3$, one of following holds:

- The curvatures are bounded and the surfaces are locally graphs over a plane.
- The curvatures blow up and the surfaces are locally double spiral staircases.

Both of these cases are illustrated by taking a sequence of rescalings of the helicoid; the first case occurs away from the axis, while the second case occurs on the axis. If we take a sequence $\Sigma_i = a_i \, \Sigma$ of rescaled helicoids where $a_i \to 0$, then the curvature blows up along the vertical axis but is bounded away from this axis. Thus, we get that:

- The intersection of the rescaled helicoids with a ball away from the vertical axis gives a collection of graphs over the plane $\{x_3 = 0\}$.
- The intersection of the rescaled helicoids with a ball centered on the vertical axis gives a double spiral staircase.

5.2. Planar domains. We next describe two main structure theorems from [**CM7**] for "nonsimply connected" embedded minimal planar domains. (Precise statements of these results and their proofs are in [**CM7**].) The first of these asserts that any such surface without small necks can be obtained by gluing together two oppositely oriented double spiral staircases. Note that when one glues two oppositely oriented double spiral staircases together, then one remains at the same level if one circles both axes. The second gives a "pair of pants" decomposition of any such surface when there are small necks, cutting the surface along a collection of short curves. After the cutting, we are left with graphical pieces that are defined over a disk with either one or two subdisks removed (a topological disk with two subdisks

removed is called a pair of pants). Both structures occur as different extremes in the two-parameter family of minimal surfaces known as Riemann examples.

When the sequence is no longer ULSC, then there are other local models for the surfaces. The simplest example is a sequence of rescaled catenoids. A sequence of rescaled catenoids converges with multiplicity two to the flat plane. The convergence is in the smooth topology except at the origin, where $|A| \to \infty$. This sequence of rescaled catenoids is not ULSC because the simple closed geodesic on the catenoid, i.e., the unit circle in the $\{x_3 = 0\}$ plane, is noncontractible, and the rescalings shrink it down to the origin. One can get other types of curvature blow-up by considering the family of embedded minimal planar domains known as the Riemann examples. Modulo translations and rotations, this family is a two-parameter family of periodic minimal surfaces, where the parameters can be thought of as the size of the necks and the angle from one fundamental domain to the next. By choosing the two parameters appropriately, one can produce sequences of Riemann examples that illustrate both structure theorems:

(1) If we take a sequence of Riemann examples where the neck size is fixed and the angles go to $\frac{\pi}{2}$, then the surfaces with angle near $\frac{\pi}{2}$ can be obtained by gluing together two oppositely-oriented double spiral staircases; see Figure 1.6. Each double spiral staircase looks like a helicoid. This sequence of Riemann examples converges to a foliation by parallel planes. The convergence is smooth away from the axes of the two helicoids (these two axes are the singular set where the curvature blows up). The sequence is ULSC since the size of the necks is fixed and thus illustrates the first structure theorem.

(2) If we take a sequence of examples where the neck sizes go to zero, then we get a sequence that is *not* ULSC. However, the surfaces can be cut along short curves into collections of graphical pairs of pants; see Figure 8.7. The short curves converge to points and the graphical pieces converge to flat planes except at these points, illustrating the second structure theorem.

We next describe the two main structure theorems for nonsimply connected embedded minimal planar domains. Each of these theorems has a compactness theorem as a consequence. The first structure theorem deals with surfaces without small necks.

Theorem 8.24 (Colding-Minicozzi, [**CM7**]). *Any nonsimply connected embedded minimal planar domain without small necks can be obtained from gluing together two oppositely oriented double spiral staircases. Moreover, if*

for some point the curvature is large, then the separation between the sheets of the double spiral staircases is small. Note that because the two double spiral staircases are oppositely oriented, then one remains at the same level if one circles both axes.

The following compactness result, or limit lamination theorem, is a consequence:

Corollary 8.25 (Colding-Minicozzi, [**CM7**]). *A ULSC (but not simply connected) sequence of embedded minimal surfaces with curvatures blowing up has a subsequence that converges smoothly to a foliation by parallel planes away from two lines. The two lines are disjoint and orthogonal to the leaves of the foliation, and the two lines are precisely the points where the curvature is blowing up.*

This corollary is similar to the case of disks, except that we get two singular curves for nondisks as opposed to just one singular curve for disks. Moreover, locally around each of the two lines the surfaces look like a helicoid around the axis, and the orientation around the two axes are opposite. The precise statement for disks is often called the *limit lamination theorem*:

Theorem 8.26 (Colding-Minicozzi, [**CM6**]). *Let $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbb{R}^3$ be a sequence of embedded minimal disks with $\partial\Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If $\sup_{B_1\cap\Sigma_i} |A|^2 \to \infty$, then there exists a subsequence, Σ_j, and a Lipschitz curve $\mathcal{S} : \mathbb{R} \to \mathbb{R}^3$ such that after a rotation of $\mathbb{R}^3$:*

1. $x_3(\mathcal{S}(t)) = t$. *(That is, $\mathcal{S}$ is a graph over the x_3-axis.)*
2. *Each Σ_j consists of exactly two multi-valued graphs away from $\mathcal{S}$ (which spiral together).*
3. *For each $1 > \alpha > 0$, $\Sigma_j \setminus \mathcal{S}$ converges in the C^α-topology to the foliation, $\mathcal{F} = \{x_3 = t\}_t$, of $\mathbb{R}^3$.*
4. $\sup_{B_r(\mathcal{S}(t))\cap\Sigma_j} |A|^2 \to \infty$ *for all $r > 0$, $t \in \mathbb{R}$.*

In [**Me2**], Meeks used [**CM3**]–[**CM6**] and [**MeR2**] to show that the singular curve $\mathcal{S}$ in the limit lamination theorem must be orthogonal to the limit planes; it follows that $\mathcal{S}$ is a straight line.

Despite the similarity of Corollary 8.25 to the case of disks, it is worth noting that the results for disks do not alone give this result. Namely, even though the ULSC sequence consists locally of disks, the compactness result for disks was in the global case where the radii go to infinity. One might wrongly assume that Corollary 8.25 could be proven by using the results for disks and a blow-up argument. However, one can construct local examples that show the difficulty of such an argument. See Chapter 1 for the local

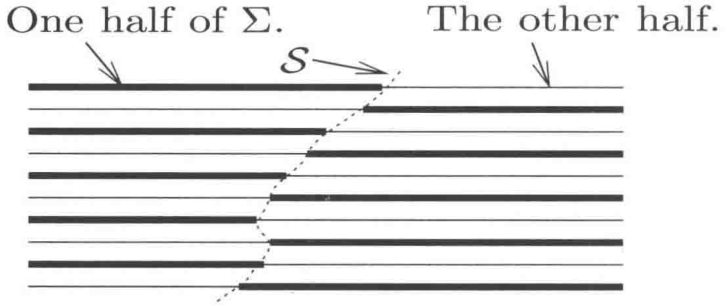

Figure 8.6. The limit lamination theorem for minimal disks. The singular set $\mathcal{S}$ and the two multi-valued graphs are shown schematically.

examples constructed in [**CM18**]; other examples can be found in [**CaL**], [**HoWh2**], [**Kh**], [**Kl**] and [**MeWe**].

The second structure theorem deals with surfaces with small necks and gives a pair of pants decomposition.

Theorem 8.27 (Colding-Minicozzi, [**CM7**]). *Any nonsimply connected embedded minimal planar domain with a small neck can be cut along a collection of short curves. After the cutting, we are left with graphical pieces that are defined over a disk with either one or two subdisks removed (a topological disk with two subdisks removed is called a pair of pants). Moreover, if for some point the curvature is large, then all of the necks are very small.*

The following compactness result is a consequence:

Corollary 8.28 (Colding-Minicozzi, [**CM7**]). *A sequence of embedded minimal planar domains that are not ULSC, but with curvatures blowing up, has a subsequence that converges to a collection of flat parallel planes.*

5.3. Uniqueness theorems.

5.3.1. *The uniqueness of the helicoid.* Using the one-sided curvature estimate and the structure theorems for embedded minimal disks of [**CM3**]–[**CM6**], Meeks Rosenberg, [**MeR2**], proved the uniqueness of the helicoid:

Theorem 8.29 (Meeks-Rosenberg, [**MeR2**]). *The plane and the helicoid are the only complete properly embedded simply connected minimal surfaces in $\mathbb{R}^3$.*

The assumption of properness in Theorem 8.29 was later removed by Colding-Minicozzi in [**CM24**]; see Section 4. Catalan proved in 1842 that any complete ruled minimal surface is either a plane or a helicoid.

5.3.2. *Genus-one helicoid: Structure results for general finite genus.* As described in Chapter 1, Weber, Hoffman and Wolf, [**HoWW**], recently constructed a new complete embedded minimal surface in $\mathbb{R}^3$. They showed

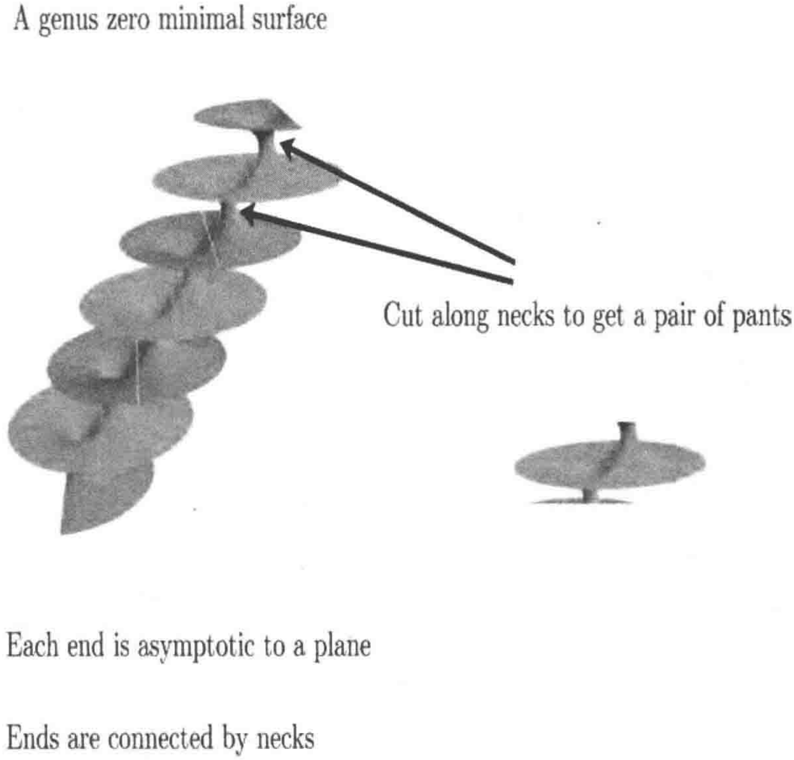

Figure 8.7. The pair of pants decomposition.
Credit: Matthias Weber, www.indiana.edu/~minimal.

that there exists a properly embedded minimal surface of genus one with a single end asymptotic to the end of the helicoid. This minimal surface is called a genus-one helicoid. Under scalings, the sequence of genus one-surfaces $\Sigma_i = a_i\,\Sigma$, where $a_i \to 0$, converges to the foliation of flat parallel planes in $\mathbb{R}^3$ just like a sequence of rescaled helicoids with curvatures blowing up. This convergence is in fact a consequence of a general result that the theorems in the previous section, stated for planar domains, also hold for sequences with fixed genus with minor changes (see [**CM7**] for the precise statements).

Using the structure results for finite genus minimal surfaces from [**CM7**], J. Bernstein and C. Breiner (cf. Meeks and Rosenberg, [**MeR2**]) proved that any complete properly embedded minimal surface with finite genus and one end must be asymptotic to the helicoid:

Theorem 8.30 (Bernstein-Breiner, [**BB2**]; cf. [**MeR2**]). *A complete properly embedded minimal surface with finite genus and one end must be asymptotic to the helicoid.*

5.3.3. *The uniqueness of the Riemann examples.* Very recently, Meeks, Perez and Ros used the structure results of [**CM3**]–[**CM7**] as well as an analytic result from [**CDM**] to prove the uniqueness of the Riemann examples:

Theorem 8.31 (Meeks-Perez-Ros, [**MePRs4**]). *The Riemann examples are the unique complete properly embedded minimal planar domains with infinitely many ends.*

Exercises

Ex 1. *Prove that the area functional on graphs is convex.*

Ex 2. *Prove that the catenoid and helicoid are minimal surfaces.*

Ex 3. *Verify that the Weierstrass representation*

$$F(z) = Re \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2} (g^{-1}(\zeta) - g(\zeta)), \frac{i}{2} (g^{-1}(\zeta) + g(\zeta)), 1 \right) \phi(\zeta),$$

where $g(z) = z$, $\phi(z) = dz/z$, $\Omega = \mathbf{C} \setminus \{0\}$ *gives a catenoid.*

Ex 4. *Derive the formulas* (1.115)–(1.117) *for the unit normal, metric, and curvature of a minimal surface from its Weierstrass data.*

Ex 5. *Compute the total curvature, i.e.,* $\int K$, *of the catenoid.*

Ex 6. *Find all rotationally symmetric solutions of the minimal surface equation on* $\mathbb{R}^2 \setminus D_1$.

Ex 7. *Prove that Enneper's surface is not embedded but its unit normal is asymptotically vertical (i.e., like a graph). What does Enneper's surface look like asymptotically?*

Ex 8. *Suppose that* M *is a three-manifold with sectional curvatures at most* -1.

- *Prove that there is no minimal immersion of* $\mathbf{S}^2$ *into* M.
- *Prove an area bound for a minimal immersion of a surface of genus* g *into* M.

Ex 9. *Suppose that* $u : D_1 \to \mathbb{R}$ *satisfies the minimal surface equation and* $|Du| \leq 1$ *on* D_1. *Use standard elliptic theory to prove a uniform bound* $|D^2u|(0) \leq C$.

Ex 10. *Find a sequence of functions* $u_j : D_1 \to \mathbb{R}$ *satisfying the minimal surface equation with* $|Du_j| \leq 1$ *on* D_1 *but* $\sup_{D_1} |D^2 u_j| \to \infty$.

Ex 11. *Show that the flux homomorphism defined in Corollary* 1.8 *determines the orientation (i.e., the direction of the axis of rotation) and the scale of a catenoid.*

Ex 12. *Fill in the details for the proof of Lemma* 1.18.

Ex 13. *Prove that any positive harmonic function on a catenoid must be constant.*

Ex 14. *Suppose that* Σ *is a minimal annulus whose boundary components are circles* $\{x^2 + y^2 = 1,\, z = H\}$ *and* $\{x^2 + y^2 = 1,\, z = -H\}$. *Prove that* H *cannot be too large.*

Ex 15. *Suppose that* Σ *is a complete minimal surface so that for all* R,

$$\mathrm{Area}(B_R \cap \Sigma) \leq 2\pi R^2 .$$

Prove that either Σ *is embedded or the union of two planes.*

Ex 16. *Define the divergence form operator*

$$Lu = \mathrm{div}(f(|\nabla u|^2)\, \nabla u)$$

for a nonnegative function f *(e.g.,* $f(s) = (1 + s)^{-1/2}$ *gives the minimal surface equation). For which functions* f *does* L *have a strong maximum principle?*

Ex 17. *Complete the proof of Theorem* 1.29 *by showing uniqueness of minimal graphs.*

Ex 18. *What is the Morse index of the catenoid?*

Ex 19. *Prove that the Morse index of the helicoid is infinite. Can you divide the helicoid into two stable pieces?*

Ex 20. *Prove Lemma* 3.24.

Ex 21. *What is the Morse index of Enneper's surface?*

Ex 22. *Classify all complete immersed minimal surfaces in* $\mathbb{R}^3$ *that have a one-to-one Gauss map.*

Ex 23. *Prove Lemma* 1.18.

Ex 24. *Let* Σ *be the unit sphere in* $\mathbb{R}^3$. *Use* (2.4) *to compute the curvature tensor* R_{ijkl}.

Ex 25. *Suppose that* $\Sigma \subset \mathbb{R}^3$ *is a minimal surface. What is the curvature of the conformal metric* $|A|^2 \langle \cdot, \cdot \rangle$*?*

Ex 26. *Suppose that* Σ *is a minimal graph and set* $v = (1 + |\nabla_{\mathbb{R}^2} u|^2)^{1/2}$ *(i.e., the inverse of the third component of the unit normal). Compute*

$$\Delta_\Sigma(|A|^2 v^2) .$$

Use this to estimate $|A|^2(0)$ *by the maximum principle.*

Ex 27. *Use scaling to prove the following "small energy estimate:" There exists* $\epsilon > 0$ *so that if* $\Delta u = -u^5$ *on* $B_R \subset \mathbb{R}^3$ *and*

$$\int_{B_R} u^6 < \epsilon ,$$

then $|u(0)| \leq R^{-1/2}$.

Ex 28. *Redo the previous exercise using integral methods (i.e., integration by parts and the Sobolev inequality) instead of scaling.*

Ex 29. *Prove that there exists* $\epsilon > 0$ *so that if* $u : \mathbb{R}^2 \to \mathbf{S}^2 \subset \mathbb{R}^3$ *satisfies the "harmonic map equation"*

$$\Delta u = -|\nabla u|^2 u$$

and has small energy

$$\int |\nabla u|^2 < \epsilon ,$$

then u *is constant.*

Ex 30. *Suppose that* $\Sigma^2 \subset \mathbf{S}^3$ *is an immersed minimal (topological)* $\mathbf{S}^2$. *Prove that* Σ *is totally geodesic (and hence "round"). Hint: Show that* A *can be regarded as a holomorphic quadratic differential.*

Ex 31. *Prove the following sharpening of Heinz's estimate: There exists a constant* C *so that if the graph of* u *is minimal over* $D_1 \subset \mathbb{R}^2$, *then the curvature* $|A|^2$ *at (the point over)* 0 *is bounded by*

$$C/(1 + |\nabla u(0)|^2) .$$

Ex 32. *Prove that the least possible area of a closed minimal surface in* $\mathbf{S}^3$ *is* 4π.

For the next three exercises, define a functional F on hypersurfaces Σ in $\mathbb{R}^{n+1}$ by

$$F(\Sigma) = \int_{\Sigma} \exp\left(-\frac{|x|^2}{4}\right) . \tag{8.97}$$

Ex 33. *Compute the first variation of* F *and the Euler-Lagrange equation for critical points of* F. *(These critical points are called self-shrinkers and come up in mean curvature flow; see* [**Hu**], [**CM25**] *and* [**CM6**] *for more details.)*

Ex 34. *Compute the second variation of* F *for a compactly supported normal variation.*

Ex 35. *Show that there are no closed hypersurfaces that are stable critical points for* F.

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